SPECIAL POINTS ON FIBERED POWERS OF ELLIPTIC SURFACES

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Abstract. Consider a fibered power of an elliptic surface. We characterize its subvarieties that contain a Zariski dense set of points that are torsion points in fibers with complex multiplication. This result can be viewed as a mix of the Manin-Mumford and André-Oort Conjecture and is related to a conjecture of Pink [18]. The main technical tool is a new height inequality. We also use it to give another proof of a case of Gubler's result on the Bogomolov Conjecture over function fields [8].

1. Introduction

In this paper we verify a combination of the Manin-Mumford and André-Oort Conjecture for a class of abelian schemes: fibered powers of an elliptic surface. The latter conjecture can also be combined with the Mordell-Lang Conjecture and we obtain results in this context. A common generalization for all three conjectures was proposed by Pink [18, 19]. An important tool in our proofs is a new height inequality on subvarieties of the ambient abelian scheme. This may be of independent interest as it generalizes to higher dimension a height theoretic result of Silverman used in the proof of his Specialization Theorem [22]. In a third application we use our height inequality to recover the Bogomolov Conjecture for products of elliptic curves over the function field of a curve.

Before stating the results we introduce the relevant class of abelian schemes. Let S be an irreducible and non-singular quasi-projective curve defined over $\overline{\mathbf{Q}}$, the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$. Let $\mathcal{E} \to S$ be an abelian scheme over S whose fibers are elliptic curves. For an integer $g \geq 1$ we let $\mathcal{A}$ denote the g-fold fibered power $\mathcal{E} \times_S \cdots \times_S \mathcal{E}$. This is also an abelian scheme over S. Let π be the structural morphism $\mathcal{A} \to S$. If $s \in S(\mathbf{C})$, it is convenient to write $\mathcal{A}_s$ for $\pi^{-1}(s)$, which is the g-th power of an elliptic curve. For reasons explained below, our results require S, and so $\mathcal{A}$, to be defined over $\overline{\mathbf{Q}}$. Nevertheless, we will speak of subvarieties of $\mathcal{A}$ defined over $\mathbf{C}$ by extending scalars without further mention.

Our main interest lies in the case where there are sufficiently many non-isomorphic abelian varieties among the fibers of $\mathcal{A} \to S$. We call $\mathcal{E}$ (or $\mathcal{A}$) isotrivial if $\mathcal{E} \to S$ becomes a constant family after a finite étale base change.

We now introduce the special points and special subvarieties of $\mathcal{A}$. We call a point in $\mathcal{A}(\mathbf{C})$ special if it is a torsion point of its respective fiber and if this fiber has complex multiplication. An irreducible closed subvariety of $\mathcal{A}$ is called special

(i) if it is an irreducible component of an algebraic subgroup of $\mathcal{A}_s$ with $s \in S(\mathbf{C})$ such that $\mathcal{A}_s$ has complex multiplication,
(ii) or if it is an irreducible component of a flat subgroup scheme of $\mathcal{A}$; cf. Section 2.4 for the definition of flat subgroup schemes.

The point of a zero-dimensional special subvariety is a special point.

An explicit and important example of an abelian scheme is the Legendre family of elliptic curves over the modular curve $Y(2) = \mathbf{P}^1 \smallsetminus \{0, 1, \infty\}$ taken as defined over $\overline{\mathbf{Q}}$. Indeed, the affine equation

$$y^2 = x(x-1)(x-\lambda)$$

determines a subvariety of $\mathbf{P}^2 \times Y(2)$ which we denote with $\mathcal{E}_L$. We let π_L denote the morphism which projects $\mathcal{E}_L$ to $Y(2)$. Then $\mathcal{E}_L$ is an abelian scheme over $Y(2)$ and the fibers of π_L are elliptic curves, cf. Section 2.1. Any elliptic curve over $\overline{\mathbf{Q}}$ has a Legendre model, so it is isomorphic to some fiber of π_L. This shows that $\mathcal{E}_L$ is not isotrivial. We write $\mathcal{A}_L$ for the g-fold fibered power of $\mathcal{E}_L$.

From a different point of view, $\mathcal{E}_L$ and $\mathcal{A}_L$ can be realized as connected mixed Shimura varieties, cf Pink's Construction 2.9 [18]. This additional structure comes with a natural notion of special points which coincides with our notion by Pink's Remark 4.13.

In an abelian variety, the Manin-Mumford Conjecture characterizes irreducible components of algebraic subgroups as those irreducible subvarieties that contain a Zariski dense set of torsion points. Its first proof is due to Raynaud [21]. The André-Oort Conjecture, on the other hand, expects special subvarieties of Shimura varieties to be precisely those irreducible subvarieties that contain a Zariski dense set of special points. Klingler and Yafaev have announced a proof [11] which assumes the Generalized Riemann Hypothesis.

Our first result characterizes subvarieties of $\mathcal{A}$ containing a Zariski dense set of special points.

Theorem 1.1. *Let $\mathcal{A}$ be as above and let us assume that $\mathcal{A}$ is not isotrivial. An irreducible closed subvariety of $\mathcal{A}$ defined over $\mathbf{C}$ contains a Zariski dense set of special points if and only if it is special.*

It follows that the Zariski closure of a set of special points in $\mathcal{A}$ is a finite union of special subvarieties.

This theorem generalizes a result of André [1, Lecture IV] which holds for curves with $\mathcal{A}$ the Legendre family of elliptic curves (so $g = 1$). Later, Pila [17] gave a proof of André's statement using a different approach.

The assumption that $\mathcal{A}$ is not isotrivial is necessary. We construct a counterexample for the constant abelian scheme $E \times \mathbf{P}^1$ where E is an elliptic curve with complex multiplication. The special subvarieties as in (i) above are of the form $E \times \{s\}$ for some $s \in \mathbf{P}^1(\mathbf{C})$; those as in (ii) are $\{P\} \times \mathbf{P}^1$ with P a torsion point of E. Let C be a curve in $E \times \mathbf{P}^1$ that is not equal to $\{P\} \times \mathbf{P}^1$ for any $P \in E(\mathbf{C})$ and not of the form $E \times \{s\}$. On considering the projection of C to E we find that our curve contains infinitely many points (P, s) with P torsion. All these points are special, but C is not.

The proof of Theorem 1.1 relies on a height inequality to be described in more detail below. Another ingredient is a finiteness statement of Poonen [20] on elliptic curves with complex multiplication and bounded Faltings height. His proof relies on results of Colmez and Nakkajima-Taguchi. We cannot work with more general abelian schemes because these results are confined to elliptic curves for the moment.

Following a suggestion of Zannier, we investigate a second "special topology" on $\mathcal{A}$ relative to a fixed elliptic curve E defined over $\mathbf{C}$. A point in $\mathcal{A}(\mathbf{C})$ is called E-special if it is a torsion point in its respective fiber and if this fiber is isogenous to E^g. Special

subvarieties of $\mathcal{A}$ are defined in a similar fashion as above. Explicitly, an irreducible closed subvariety of $\mathcal{A}$ defined over $\mathbf{C}$ is called E-special

(i) if it is an irreducible component of an algebraic subgroup of $\mathcal{A}_s$ with $s \in S(\mathbf{C})$ such that $\mathcal{E}_s$ is isogenous to E,
(ii) or if it is an irreducible component of a flat subgroup scheme of $\mathcal{A}$.

The point of an zero-dimensional E-special subvariety is E-special.

The set of E-special points of $\mathcal{A}_L$ is a Hecke orbit, as defined in Section 3 [18], of the zero element of an appropriate fiber, cf. the proof of Proposition 5.1 there.

Theorem 1.2. *Let $\mathcal{A}$ be as above and let us assume that $\mathcal{A}$ is not isotrivial. Let E be an elliptic curve defined over $\overline{\mathbf{Q}}$. An irreducible closed subvariety of $\mathcal{A}$ defined over $\mathbf{C}$ contains a Zariski dense set of E-special points if and only if it is E-special.*

So, the Zariski closure of a set of E-special points in $\mathcal{A}$ is a finite union of E-special subvarieties.

In addition to the height inequality which is also used in Theorem 1.1, the theorem above relies on a result of Szpiro and Ullmo [25]. They describe the distribution of the Faltings height in a fixed isogeny class of elliptic curves without complex multiplication. As was the case with the finiteness statement of Poonen, a version of this result for more general abelian varieties would be needed to treat abelian schemes with more general fibers.

The main technical tool in the proofs of Theorems 1.1 and 1.2 is a height inequality on subvarieties of $\mathcal{A}$ given in Theorem 1.3. It relates the restrictions of two different height functions on $\mathcal{A}$ to a fixed subvariety. Since our heights are only defined when dealing with algebraic points we shall assume that S and E are defined over $\overline{\mathbf{Q}}$.

The first height function is derived from a height on S; it measures the corresponding fiber in $\mathcal{A}$. We may assume that S is a Zariski open subset of an irreducible and non-singular projective curve $\overline{S}$ over $\overline{\mathbf{Q}}$. On $\overline{S}$ we fix a line bundle $\mathcal{L}$. Given this pair we may choose a height function $h_{\overline{S},\mathcal{L}} : \overline{S}(\overline{\mathbf{Q}}) \to \mathbf{R}$, cf. Section 2.1. The first height is just the composition $h_{\overline{S},\mathcal{L}} \circ \pi : \mathcal{A}(\overline{\mathbf{Q}}) \to \mathbf{R}$.

The second height function $\hat{h}_{\mathcal{A}} : \mathcal{A}(\overline{\mathbf{Q}}) \to [0,\infty)$ is more closely related to the group structure on the fibers of $\mathcal{A} \to S$ above points in $S(\overline{\mathbf{Q}})$. Indeed, for any $s \in S(\overline{\mathbf{Q}})$ the fiber $\mathcal{A}_s$ is the g-th power of an elliptic curve. It is equipped with the so-called Néron-Tate height $\mathcal{A}_s(\overline{\mathbf{Q}}) \to [0,\infty)$ which we describe more thoroughly in Section 2.1. Letting s vary over $S(\overline{\mathbf{Q}})$ we obtain a Néron-Tate height $\hat{h}_{\mathcal{A}} : \mathcal{A}(\overline{\mathbf{Q}}) \to [0,\infty)$.

Our two height functions are unrelated in the following sense. It is not difficult to construct an infinite sequence of points $P_1, P_2, \ldots \in \mathcal{A}(\overline{\mathbf{Q}})$ such that $\hat{h}_{\mathcal{A}}(P_k)$ is constant and $h_{\overline{S},\mathcal{L}}(\pi(P_k))$ unbounded. For example, it suffices to take P_k any torsion point in $\mathcal{A}_{\pi(P)}$ and the sequence $\pi(P_k)$ of unbounded height. Then $\hat{h}_{\mathcal{A}}(P_k) = 0$ since the Néron-Tate height vanishes on torsion points.

If $\mathcal{A}$ is not isotrivial, the situation changes when our points lie on an irreducible subvariety $X \subset \mathcal{A}$ which is not "special" in a slightly weaker sense than above. This is the content of the height inequality in Theorem 1.3. We will bound $h_{\overline{S},\mathcal{L}} \circ \pi$ from above linearly in terms of $\hat{h}_{\mathcal{A}}$ when restricted to a certain natural Zariski open and non-empty subset of X. We define this subset now.

For an irreducible closed subvariety $X \subset \mathcal{A}$ defined over $\mathbf{C}$ we set

$$X^\star = X \smallsetminus \bigcup_Z Z$$

where Z runs over all closed subvarieties of X that are irreducible components of flat subgroup schemes of $\mathcal{A}$. The fact that $X^\star$ is Zariski open is not immediately obvious since the union may be infinite. But it is part of the theorem below and will follow from the Manin-Mumford Conjecture applied to the generic fiber of $\mathcal{A} \to S$. We will also obtain a necessary and sufficient condition for the non-emptiness of $X^\star$.

Theorem 1.3. *Let $\mathcal{A}$ be as above and let $X \subset \mathcal{A}$ be an irreducible closed subvariety defined over $\mathbf{C}$.*

(i) *The set $X^\star$ is Zariski open in X. It is empty if and only if X is an irreducible component of a flat subgroup scheme of $\mathcal{A}$.*

(ii) *If X is defined over $\overline{\mathbf{Q}}$ and if $\mathcal{A}$ is not isotrivial there exists a constant $c > 0$ such that*

$$h_{\overline{S},\mathcal{L}}(\pi(P)) \le c \max\{1, \hat{h}_{\mathcal{A}}(P)\} \quad \textit{for all} \quad P \in X^\star(\overline{\mathbf{Q}}). \tag{1.1}$$

If X is a curve, then Theorem 1.3(ii) can be proved using Silverman's Theorem B [22]; we state it as Theorem 6.1 below. In fact, Silverman's result provides a more precise estimate for more general abelian schemes. His proof depends on the fact that an irreducible projective curve has infinite cyclic Néron-Severi group. The advantage of our theorem is that it can handle subvarieties of arbitrary dimension.

Masser and Zannier [15] proved that a certain explicit curve in $\mathcal{E}_L \times_{Y(2)} \mathcal{E}_L$ contains only finitely many points which are torsion in their respective fibers. Their result is also related to Pink's general conjecture [19]. One step in their argument required a height bound as in Theorem 1.3 for curves. For this they used Silverman's result mentioned further up. One could hope that our height theoretic result may play a role in a generalization of Masser and Zannier's result to higher dimensional subvarieties.

The particular abelian scheme $\mathcal{A}_L \to Y(2)$ defined using the Legendre family plays a central role in the proof of Theorem 1.3. By adding level structure to $\mathcal{E} \to S$ we will be able to reduce the proof to the case $\mathcal{A} = \mathcal{A}_L$ and $S = Y(2)$. This allows us to exploit the very explicit nature of the Legendre family.

We briefly sketch the lines of the proof in this setting. Perhaps surprisingly, the basic strategy is to construct sufficiently many points on X which are torsion in their respective fibers. This is done in Proposition 3.1 if X is a hypersurface. The existence of many such torsion points has implications for a certain intersection number on an appropriate compactification of X. This information will be used to establish the existence of an auxiliary non-zero global section of a certain line bundle. By arguments from height theory this global section is ultimately responsible for the inequality in Theorem 1.3(ii). If X is not a hypersurface, then we will apply an inductive argument.

There is an implicit restriction on X in part (ii) of the theorem above. Namely, $X^\star \neq \emptyset$ since the statement is trivial otherwise. This suggests that there must be an obstruction in the sketch above. Indeed, it is the argument in Proposition 3.1 which may fail if X is an irreducible component of a flat subgroup scheme. Part of the proof of this proposition concerns the Zariski denseness of what one might call an analytic subgroup scheme of $\mathcal{A}_L$. This is done by studying the local monodromy of our abelian scheme around the

cusps 0 and 1 of $Y(2)$. Roughly speaking, monodromy allows us to extract information from the hypothesis $X^\star \neq \emptyset$. For abelian schemes, local monodromy is known to be quasi-unipotent. For our specific $\mathcal{A}_L$ we will see that it is even unipotent around 0 and 1. We will show that the nilpotent part has sufficiently large rank. This allows us to apply Kronecker's Theorem from diophantine approximation giving the argument an ergodic flavor.

Our claim on Zariski denseness can be rephrased by saying that a certain set of functions is algebraically independent over the field $\mathbf{C}(\lambda)$. These functions turn out closely related to solutions of type VI Painlevé differential equations. In the setting we consider, they are known to be transcendental over $\mathbf{C}(\lambda)$. But algebraic independence seems to be new.

Local monodromy of an abelian scheme over a projective base is of finite order. Hence, the nilpotent part is trivial. It would be interesting to see if and how our approach can adapt to abelian schemes over curves lacking cusps.

We come to a final application of the height inequality. The Bogomolov Conjecture for abelian varieties defined over a number field generalizes the Manin-Mumford Conjecture. Whereas the Manin-Mumford Conjecture describes the distribution of torsion points on subvarieties of abelian varieties, the Bogomolov Conjecture governs those points which merely have small Néron-Tate height.

Over number fields, the Bogomolov Conjecture is a theorem due to the work of Ullmo and Zhang. It has an analog for abelian varieties defined over function fields since one can also define the Néron-Tate height in this setting. The Bogomolov Conjecture is open in the context of function fields. But Gubler [8] has made important progress by proving it if the abelian variety is totally degenerate at one place of the function field.

We prove the Bogomolov Conjecture for the power of an elliptic curve defined over the function field of a curve and with non-constant j-invariant. This abelian variety can be realized as the generic fiber of some $\mathcal{A} \to S$ as in Theorem 1.3. Here our height inequality comes into the picture; we shall combine it with the more precise statement of Silverman which holds for curves. In fact this particular case of the Bogomolov Conjecture is covered by Gubler's work. But our approach differs from his and provides another approach to this problem.

Let K be the function field of an irreducible non-singular projective curve defined over $\overline{\mathbf{Q}}$ and let E be an elliptic curve defined over K. The j-invariant of E is an element of K; we call it non-constant if it lies in $K \smallsetminus \overline{\mathbf{Q}}$. Let $\overline{K}$ be an algebraic closure of K. We fix an ample and symmetric line bundle on E. This induces a Néron-Tate height function $\hat{h} : E^g(\overline{K}) \to [0, \infty)$, see Section 6 for references in the function field setting.

Theorem 1.4. *Let $K, \overline{K}, E$, and $\hat{h}$ be as above. We shall assume that the j-invariant of E is non-constant. Let $X \subset E^g$ be an irreducible closed subvariety defined over $\overline{K}$ which is not an irreducible component of an algebraic subgroup of E^g. There exist $\epsilon > 0$ and a Zariski closed proper subset $Z \subset X$ such that $P \in (X \smallsetminus Z)(\overline{K})$ implies $\hat{h}(P) \geq \epsilon$.*

The article is organized as follows. In Section 2 we introduce much of the notation used throughout later sections. Section 3 contains the estimate on counting torsion points alluded to in the sketch above. We will use it in Section 4 to prove a preliminary height inequality. In Section 5 we then deduce Theorem 1.3. In the same section we also prove Theorems 1.1 and 1.2. Finally, Theorem 1.4 is shown in Section 6.

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2. Preliminaries

2.1. **One Abelian Scheme with Two Heights.** Let $Y(2) = \mathbf{P}^1 \smallsetminus \{0, 1, \infty\}$ and let $\mathcal{E}_L$ be the closed subvariety of $\mathbf{P}^2 \times Y(2)$ given by

$$\Big\{([x : y : z], [\lambda : 1]);\ zy^2 = x(x - z)(x - z\lambda)\Big\} \subset \mathbf{P}^2 \times Y(2).$$

We let $\pi_L = \mathcal{E}_L \to Y(2)$ denote the projection onto the second factor. Each fiber of π_L is an elliptic curve given in Legendre form. The zero section $\epsilon_L : Y(2) \to \mathcal{E}_L$ is defined as $\epsilon_L(\lambda) = ([0 : 1 : 0], \lambda)$; it is a closed immersion. The addition morphism on each fiber of $\mathcal{E}_L \to Y(2)$ extends to a morphism $\mathcal{E}_L \times_{Y(2)} \mathcal{E}_L \to \mathcal{E}_L$. Similarly, we have a morphism $\mathcal{E}_L \to \mathcal{E}_L$ which is fiberwise the inversion. Therefore $\mathcal{E}$ is a group scheme over $Y(2)$. The morphism π_L is proper because it is a composition of the closed immersion $\mathcal{E}_L \hookrightarrow \mathbf{P}^2 \times Y(2)$ and the projection morphism $\mathbf{P}^2 \times Y(2) \to Y(2)$ which is proper. The morphism π_L is smooth and geometrically connected because its fibers are elliptic curves. Hence $\mathcal{E}_L$ is an abelian scheme over $Y(2)$.

Throughout this paper $g \geq 1$ is an integer and $\mathcal{A}_L = \mathcal{E}_L \times_{Y(2)} \cdots \times_{Y(2)} \mathcal{E}_L$ is the g-fold fibered power of $\mathcal{E}_L$ over $Y(2)$. It is an abelian scheme over $Y(2)$ of dimension $g+1$. We have a natural embedding $\mathcal{A}_L \subset (\mathbf{P}^2)^g \times Y(2)$. By abuse of notation we write π_L for the canonical map $\mathcal{A}_L \to Y(2)$ and $\epsilon_L : Y(2) \to \mathcal{A}_L$ for the zero section.

We consider the absolute logarithmic Weil height $h : \mathbf{P}^n(\overline{\mathbf{Q}}) \to [0, \infty)$ and sometimes call it the projective height. For a definition and some basic properties we refer to Chapter 1.5 [2]. Since we have a natural inclusion $Y(2) \subset \mathbf{P}^1$, the projective height restricts to a height $h : Y(2)(\overline{\mathbf{Q}}) \to [0, \infty)$. The Weil height of an algebraic number x is the projective height of $[x : 1] \in \mathbf{P}^1(\overline{\mathbf{Q}})$.

Say Z is a projective variety defined over $\overline{\mathbf{Q}}$ and $\mathcal{L}$ a line bundle on Z. This pair determines an equivalence class $h_{Z,\mathcal{L}}$ of real valued functions $Z(\overline{\mathbf{Q}}) \to \mathbf{R}$ where two functions are taken to be equivalent if the absolute value of their difference is uniformly bounded from above. The association $(Z, \mathcal{L}) \mapsto h_{Z,\mathcal{L}}$ has useful functorial properties which we use freely throughout this paper. For more information on these and a construction we refer to Chapter 2 of Bombieri and Gubler's book [2]. It is sometimes convenient to use the same symbol $h_{Z,\mathcal{L}}$ for a specific representative in the equivalence class. We will point out such a choice.

We discuss two notions for the height of a point $P = (P_1, \ldots, P_g, \pi_L(P)) \in \mathcal{A}_L(\overline{\mathbf{Q}})$. As in the introduction we could use $h(\pi_L(P))$ to gauge the fiber containing P. It is sometimes more convenient to work with "total height" given by

$$h_{\mathcal{A}_L}(P) = h(P_1) + \cdots + h(P_g) + h(\pi_L(P)), \tag{2.1}$$

we recall $P_1, \ldots, P_g \in \mathbf{P}^2(\overline{\mathbf{Q}})$.

Let E be any elliptic curve defined over $\overline{\mathbf{Q}}$. The zero element of E considered as a Weil divisor determines a line bundle $\mathcal{L}$ on E. There is a rational function x on E whose

only pole is at the zero element and of order two there. Then x extends to a morphism $E \to \mathbf{P}^1$ and a valid choice of representative for $h_{E,\mathcal{L}}$ is $\frac{1}{2}h \circ x : E(\overline{\mathbf{Q}}) \to [0,\infty)$ with h the projective height. Tate's Limit Argument, cf. Chapter 9.2 [2], enables us to choose a canonical element in the equivalence class $h_{E,\mathcal{L}}$. We let $\hat{h}_E : E(\overline{\mathbf{Q}}) \to [0,\infty)$ denote this element and call it the Néron-Tate height. The Néron-Tate height has the advantage that if E' is an elliptic curve over $\overline{\mathbf{Q}}$ and $f : E \to E'$ is an isomorphism of elliptic curves, then functorial properties of the height imply $\hat{h}_{E'}(f(P)) = \hat{h}_E(P)$ for all $P \in E(\overline{\mathbf{Q}})$.

Let $\mathcal{A}$ and S be as in the introduction. If $s \in S(\overline{\mathbf{Q}})$ and $P = (P_1,\ldots,P_g) \in \mathcal{E}_s^g(\overline{\mathbf{Q}})$, we set

$$\hat{h}_{\mathcal{A}}(P) = \hat{h}_{\mathcal{E}_s}(P_1) + \cdots + \hat{h}_{\mathcal{E}_s}(P_g) \geq 0 \tag{2.2}$$

and call this the Néron-Tate height on $\mathcal{A}$. Of course, this also determines a Néron-Tate height on $\mathcal{A}_L$.

2.2. **Period Map.** For $\tau \in \mathbf{H}$, where $\mathbf{H} \subset \mathbf{C}$ is the upper half-plane, we have the Weierstrass function

$$\wp(z;\tau) : \mathbf{C} \smallsetminus (\mathbf{Z} + \tau\mathbf{Z}) \to \mathbf{C},$$

which is holomorphic on its domain and $\mathbf{Z}+\tau\mathbf{Z}$-periodic; a reference is Chapter 1 [12]. If $\tau,\tau' \in \mathbf{H}$ generate the same lattice, i.e. $\mathbf{Z}+\tau\mathbf{Z} = \mathbf{Z}+\tau'\mathbf{Z}$, then $\wp(\cdot;\tau) = \wp(\cdot;\tau')$. We recall the classical equalities

$$\wp(\alpha z;\alpha\tau) = \alpha^{-2}\wp(z;\tau), \quad \text{and} \quad \wp'(\alpha z;\alpha\tau) = \alpha^{-3}\wp(z;\tau) \tag{2.3}$$

which hold if the corresponding expressions are well-defined. The Weierstrass function and its derivative satisfy the differential equation

$$\wp'(z;\tau)^2 = 4(\wp(z;\tau) - e_1(\tau))(\wp(z;\tau) - e_2(\tau))(\wp(z;\tau) - e_3(\tau))$$

where

$$e_1(\tau) = \wp(\tau/2;\tau), \quad e_2(\tau) = \wp(1/2;\tau), \quad \text{and} \quad e_3(\tau) = \wp((1+\tau)/2;\tau)$$

are pairwise distinct complex numbers for fixed τ. Thus

$$\Lambda(\tau) = \frac{e_3(\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)},$$

is a well-defined holomorphic map $\mathbf{H} \to \mathbf{C} \smallsetminus \{0,1\} = Y(2)(\mathbf{C})$.

We now exhibit a local inverse for Λ. Gauss's hypergeometric function

$$F(\lambda) = {}_2F_1\left(\frac{1}{2},\frac{1}{2},1,\lambda\right) = \sum_{n=0}^{\infty} \frac{(2n)!^2}{2^{4n}n!^4}\lambda^n$$

is holomorphic on the open unit disc in $\mathbf{C}$. We set

$$\omega_1(\lambda) = F(\lambda)\pi \quad \text{and} \quad \omega_2(\lambda) = F(1-\lambda)\pi i$$

and obtain two functions, both holomorphic on

$$\Sigma = \{\lambda \in \mathbf{C};\ |\lambda| < 1 \text{ and } |1-\lambda| < 1\}.$$

By Theorem 6.1, page 184 [10] the complex numbers $\omega_1(\lambda), \omega_2(\lambda)$ are periods of an elliptic curve for $\lambda \in \Sigma$. So they are $\mathbf{R}$-linearly independent and in particular, $\omega_1(\lambda) \neq 0$. We obtain a holomorphic map $T : \Sigma \to \mathbf{C}$ defined by

$$T(\lambda) = \frac{\omega_2(\lambda)}{\omega_1(\lambda)} = \frac{F(1-\lambda)}{F(\lambda)} i. \tag{2.4}$$

Since $T(1/2) = i$ lies in $\mathbf{H}$ we have $T(\Sigma) \subset \mathbf{H}$.

Lemma 2.1. (i) *For any $\lambda \in \Sigma$ we have $\Lambda(T(\lambda)) = \lambda$.*

(ii) *Let $\tau \in \mathbf{H}$, we have*

$$e_k(\tau + 2) = e_k(\tau) \quad \textit{and} \quad e_k\left(\frac{\tau}{-2\tau+1}\right) = (-2\tau+1)^2 e_k(\tau) \quad \textit{for} \quad 1 \leq k \leq 3 \tag{2.5}$$

as well as

$$\Lambda(\tau + 2) = \Lambda(\tau) \quad \textit{and} \quad \Lambda\left(\frac{\tau}{-2\tau+1}\right) = \Lambda(\tau). \tag{2.6}$$

Proof. Let $\lambda \in \Sigma$ and $\tau = T(\lambda)$. It follows from Theorem 6.1, page 184 [10] that $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ and $(\mathcal{E}_L)_\lambda(\mathbf{C})$ are isomorphic complex tori.

We have a holomorphic map given by

$$z \mapsto \left[\frac{\wp(z;\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)} : \frac{\wp'(z;\tau)}{2(e_2(\tau) - e_1(\tau))^{3/2}} : 1\right]$$

if $z \in \mathbf{C} \smallsetminus (\mathbf{Z} + \tau\mathbf{Z})$ and $z \mapsto [0 : 1 : 0]$ for $z \in \mathbf{Z} + \tau\mathbf{Z}$; the choice of root is irrelevant. A straightforward calculation shows that the image of this holomorphic map lies in $(\mathcal{E}_L)_{\Lambda(\tau)}$. It is classical, that this map induces an isomorphism of complex tori between $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$ and $(\mathcal{E}_L)_{\Lambda(\tau)}(\mathbf{C})$. Hence the j-invariants of $(\mathcal{E}_L)_\lambda$ and $(\mathcal{E}_L)_{\Lambda(\tau)}$ are equal. In other words,

$$2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} = 2^8 \frac{(\Lambda(\tau)^2 - \Lambda(\tau) + 1)^3}{\Lambda(\tau)^2(\Lambda(\tau) - 1)^2}, \tag{2.7}$$

by Remark 1.4, page 87 [10]. This equality implies

$$\Lambda(\tau) \in \left\{\lambda, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{\lambda-1}, \frac{\lambda-1}{\lambda}, 1-\lambda\right\}. \tag{2.8}$$

Since $\Lambda \circ T$ is analytic, it suffices to show $\Lambda(\tau) = \lambda$ for $\tau = T(\lambda)$ and all sufficiently small $\lambda \in (0, 1/2)$ in order to deduce part (i). The j-invariant j of $(\mathcal{E}_L)_\lambda$ is in $(1728, \infty)$ by (2.7). So there is $x \geq 1$ such that τ is equivalent to ix under the usual action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{H}$. It follows from (2.4) and $\lambda \in \mathbf{R}$ that τ has real part 0. Moreover, $\lambda < 1/2 < 1 - \lambda$ and so τ has imaginary part at least 1 since F increases on $(0, 1)$. In particular, τ is already in the usual fundamental domain of the action of $\mathrm{SL}_2(\mathbf{Z})$ on $\mathbf{H}$, hence $\tau = ix$. Remark 2, page 251 [12] gives

$$\Lambda(\tau) = \Lambda(ix) = 16e^{-\pi x} \prod_{n=1}^{\infty} \left(\frac{1 + e^{-2\pi x n}}{1 + e^{-2\pi x(n-1/2)}}\right)^8,$$

so $\Lambda(\tau) > 0$. Moreover, $\Lambda(\tau) < 16e^{-\pi x}$ because each factor in the infinite product above is in $(0, 1)$. As λ approaches 0, the j-invariant j goes to $+\infty$. But j is the value of the modular j-function at ix; properties of this function imply that $x \to +\infty$ as $j \to +\infty$.

Therefore $\Lambda(\tau) \in (0, 1/2)$ for λ sufficiently small. By (2.8) the only possibility for $\Lambda(\tau)$ is λ. This concludes the proof of (i).

For the proof of part (ii) we remark that (2.5) implies (2.6) by definition of Λ. By periodicity of the Weierstrass function we get

$$\begin{aligned} e_1(\tau+2) &= \wp(\tau/2+1;\tau+2) &&= e_1(\tau), \\ e_2(\tau+2) &= \wp(1/2;\tau+2) &&= e_2(\tau), \quad \text{and} \\ e_3(\tau+2) &= \wp(1/2+\tau/2+1;\tau+2) &&= e_3(\tau). \end{aligned}$$

So the first equality in (2.5) holds for all k.

Using (2.3) we derive in a similar way as above that the second equality in (2.5) holds for all k. $\square$

We define the local period map $\Omega : \Sigma \to \mathrm{Mat}_{g,2g}(\mathbf{C})$ as

$$\Omega(\lambda) = \begin{bmatrix} \omega_1(\lambda) & \omega_2(\lambda) & & & \\ & & \ddots & & \\ & & & \omega_1(\lambda) & \omega_2(\lambda) \end{bmatrix}. \tag{2.9}$$

2.3. **Exponential Map.** We take some time to introduce the (local) exponential map of the Legendre family which will prove useful later on.

By Remark 2, page 251 [12] we may write

$$e_2(\tau) - e_1(\tau) = r(\tau)^2 \quad \text{where} \quad r(\tau) = \pi \prod_{n\geq 1}(1-e^{2\pi i n\tau})^2(1+e^{2\pi i(n-1/2)\tau})^4 \tag{2.10}$$

for any $\tau \in \mathbf{H}$. The map $r : \mathbf{H} \to \mathbf{C}$ is holomorphic and non-vanishing

We obtain a holomorphic map $\exp : \mathbf{C} \times \Sigma \to \mathbf{P}^2(\mathbf{C}) \times \Sigma$, the exponential map, given by

$$(z,\lambda) \mapsto \left(\left[\frac{\wp(z/\omega_1(\lambda); T(\lambda)) - e_1(T(\lambda))}{e_2(T(\lambda)) - e_1(T(\lambda))} : \frac{\wp'(z/\omega_1(\lambda); T(\lambda))}{2r(T(\lambda))^3} : 1 \right], \lambda \right)$$

if $z \notin \mathbf{Z} + T(\lambda)\mathbf{Z}$ and $\exp(z,\lambda) = ([0:1:0],\lambda)$ else wise.

The next lemma summarizes some basic facts about the exponential map.

Lemma 2.2. (i) *The diagram*

$$\begin{tikzcd} \mathbf{C} \times \Sigma \arrow[rr, "\exp"] \arrow[dr] & & \mathcal{E}_L(\mathbf{C}) \arrow[dl] \\ & Y(2)(\mathbf{C}) & \end{tikzcd} \tag{2.11}$$

commutes; the vertical arrows are projections and the bottom arrow is the inclusion.

(ii) *For fixed $\lambda \in Y(2)(\mathbf{C})$ the map $\mathbf{C} \to (\mathcal{E}_L)_\lambda(\mathbf{C})$ given by $z \mapsto \exp(z,\lambda)$ is a surjective group homomorphism with kernel $\mathbf{Z} + T(\lambda)\mathbf{Z}$.*

Proof. A straightforward calculation shows that $\exp(z,\lambda) \in (\mathcal{E}_L)_{\Lambda(T(\lambda))}(\mathbf{C})$ for $z \in \mathbf{C}$ and $\lambda \in \Sigma$. We already know $\Lambda(T(\lambda)) = \lambda$ from Lemma 2.1(i), so the diagram in (2.11) commutes. Part (ii) is classical. $\square$

By abuse of notation we write

$$\exp : \mathbf{C}^g \times \Sigma \to (\mathcal{A}_L)_\Sigma$$

for the fibered product of the exponential map, here $(\mathcal{A}_L)_\Sigma = {\pi_L}^{-1}(\Sigma) \subset \mathcal{A}_L(\mathbf{C})$.

Let $\xi = (\xi_1, \xi_2) \in (\mathbf{R}/\mathbf{Z})^2$. We define a holomorphic map $\widetilde{\rho}_\xi : \mathbf{H} \to \mathcal{E}_L(\mathbf{C})$ which is needed in Section 3. If $\xi = 0$ we set $\widetilde{\rho}_\xi(\tau) = ([0:1:0], \Lambda(\tau))$. If $\xi \neq 0$, then we set

$$\widetilde{\rho}_\xi(\tau) = \left(\left[\frac{\wp(\xi_1 + \tau\xi_2; \tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)} : \frac{\wp'(\xi_1 + \tau\xi_2; \tau)}{2r(\tau)^3} : 1 \right], \Lambda(\tau) \right),$$

this map is well-defined by periodicity of the Weierstrass function. We remark, that $\widetilde{\rho}_\xi(\tau) \in (\mathcal{E}_L)_{\Lambda(\tau)}(\mathbf{C})$.

If $\xi = (\xi_1, \ldots, \xi_{2g}) \in (\mathbf{R}/\mathbf{Z})^{2g}$, then the g-fold product of $\widetilde{\rho}_{(\xi_1,\xi_2)}, \ldots, \widetilde{\rho}_{(\xi_{2g-1},\xi_{2g})}$ is a holomorphic map $\mathbf{H} \to \mathcal{A}_L(\mathbf{C})$, which by abuse of notation we also call $\widetilde{\rho}_\xi$. So $\widetilde{\rho}_\xi(\tau) \in (\mathcal{A}_L)_{\Lambda(\tau)}(\mathbf{C})$.

Lemma 2.3. *Let $\xi = (\xi_1, \ldots, \xi_{2g}) \in (\mathbf{R}/\mathbf{Z})^{2g}$.*

(i) *If $\lambda \in \Sigma$, then $\exp(\Omega(\lambda)\xi^{\mathtt{T}}, \lambda)$ is well-defined and*

$$\widetilde{\rho}_\xi(T(\lambda)) = \exp(\Omega(\lambda)\xi^{\mathtt{T}}, \lambda),$$

where $^{\mathtt{T}}$ means transpose.

(ii) *[Local monodromy around 0.] If $\tau \in \mathbf{H}$, then*

$$\widetilde{\rho}_\xi(\tau + 2) = \widetilde{\rho}_{\xi + 2(\xi_2, 0, \xi_4, 0, \ldots, \xi_{2g}, 0)}(\tau).$$

(iii) *[Local monodromy around 1.] If $\tau \in \mathbf{H}$, then*

$$\widetilde{\rho}_\xi\left(\frac{\tau}{-4\tau + 1}\right) = \widetilde{\rho}_{\xi - 4(0, \xi_1, 0, \xi_3, \ldots, 0, \xi_{2g-1})}(\tau).$$

Proof. Part (i) follows from the definition of the exponential map, $\widetilde{\rho}_\xi$, the period matrix $\Omega(\lambda)$, and from the equality $\Lambda(T(\lambda)) = \lambda$.

A direct consequence of the left-hand side of (2.10) is $r(\tau + 2) = r(\tau)$. Part (ii) follows from this, from the first equalities in (2.5) and (2.6), and from periodicity of the Weierstrass function.

We turn to part (iii). Say $(\xi_1, \xi_2) \in (\mathbf{R}/\mathbf{Z})^2 \smallsetminus \{0\}$ and let $\tau' = \tau/(-2\tau + 1)$ and $\tau'' = \tau'/(-2\tau' + 1) = \tau/(-4\tau + 1)$. The functional equalities (2.3) imply

$$\wp\left(\xi_1 + \xi_2\tau''; \tau''\right) = (-4\tau + 1)^2 \wp(\xi_1 + \tau(-4\xi_1 + \xi_2); \tau)$$

and

$$\wp'\left(\xi_1 + \xi_2\tau''; \tau''\right) = (-4\tau + 1)^3 \wp'(\xi_1 + \tau(-4\xi_1 + \xi_2); \tau).$$

Moreover, $e_{1,2}(\tau') = (-2\tau + 1)^2 e_{1,2}(\tau)$ by (2.5). Using these, the left-hand side in (2.10) implies $r(\tau') = \chi(\tau)(-2\tau + 1)r(\tau)$ with $\chi(\tau) \in \{\pm 1\}$. But χ is continuous on the connected space $\mathbf{H}$, so it is a constant χ. We derive $r(\tau'') = \chi(-2\tau' + 1)r(\tau') = \chi^2(-4\tau + 1)r(\tau) = (-4\tau + 1)r(\tau)$. Finally, $\Lambda(\tau'') = \Lambda(\tau') = \Lambda(\tau)$ by (2.6). Hence $\widetilde{\rho}_{(\xi_1,\xi_2)}(\tau'') = \widetilde{\rho}_{(\xi_1, -4\xi_1 + \xi_2)}(\tau)$. This equality also holds if $(\xi_1, \xi_2) = 0$. Part (iii) follows since if ξ is as in the hypothesis, then $\widetilde{\rho}_\xi$ is the product of $\widetilde{\rho}_{(\xi_1,\xi_2)}, \ldots, \widetilde{\rho}_{(\xi_{2g-1},\xi_{2g})}$. $\square$

2.4. **Flat Subgroup Schemes of $\mathcal{A}$.** In this section we let $\mathcal{E} \to S$ be as in the introduction. That is, $\mathcal{E}$ is an abelian scheme over a non-singular and irreducible quasi-projective curve S defined over $\overline{\mathbf{Q}}$. Moreover, the fibers of $\mathcal{E} \to S$ are elliptic curves. We let $\mathcal{A}$ denote a g-th fibered power of $\mathcal{E}$ over S with π the structural morphism $\mathcal{A} \to S$.

We call a possibly reducible closed subvariety $H \subset \mathcal{A}$ a subgroup scheme if it contains the image of the zero section $S \to \mathcal{A}$, if it is mapped to itself by the inversion $\mathcal{A} \to \mathcal{A}$ morphism, and such that the image of $H \times_S H$ under the addition morphism $\mathcal{A} \times_S \mathcal{A} \to \mathcal{A}$ is in H. In this article we disregard standard terminology and required subgroup schemes to be reduced. This is justified since the base S is a curve over a field of characteristic 0.

A subgroup scheme may fail to be flat over S; it can have horizontal as well as vertical fibers. We call a subgroup scheme H of $\mathcal{A}$ flat if its irreducible components dominate S. By Proposition III 9.7 [9] this amounts to saying that $\pi|_H : H \to S$ is flat.

For an integer N we have the multiplication-by-N morphism $[N] : \mathcal{A} \to \mathcal{A}$. It is proper since $\pi = \pi \circ [N]$ is proper. If $s \in S(\mathbf{C})$ and if $X \subset \mathcal{A}$ is Zariski closed, then X_s denotes the Zariski closed set $\pi|_X^{-1}(s) \subset \mathcal{A}_s$.

The generic fiber of $\mathcal{E} \to S$ is an elliptic curve over $\overline{\mathbf{Q}}(S)$, the function field of S. Its j-invariant is an element of $\overline{\mathbf{Q}}(S)$. It extends to a morphism $j : S \to Y(1)$ with $j(s) \in \mathbf{C}$ the j-invariant of the elliptic curve $\mathcal{E}_s$ for all $s \in S(\mathbf{C})$.

Lemma 2.4. *If $\mathcal{A}$ is not isotrivial, then j is non-constant.*

Proof. We consider the j-invariant as an element of $\overline{\mathbf{Q}}(S)$ and assume it to be constant. The generic fiber of $\mathcal{E} \to S$ is an elliptic curve with j-invariant in $\overline{\mathbf{Q}}$. Hence it is isomorphic, over some finite field extension K of $\overline{\mathbf{Q}}(S)$, to the base change to K of an elliptic curve defined over $\overline{\mathbf{Q}}$. The lemma follows from Theorem 3.1(i) [26]. □

Below we give a description of all subgroup schemes of the abelian scheme $\mathcal{A}$.

Any $\varphi = (a_1, \ldots, a_g) \in \mathbf{Z}^g$ induces a morphism $\varphi : \mathcal{A} \to \mathcal{E}$ with $\varphi(P_1, \ldots, P_g) = [a_1](P_1) + \cdots + [a_g](P_g)$. Then φ is proper since $\pi = \pi \circ \varphi$ is proper. From now on we identify elements of $\mathbf{Z}^g$ with the associated morphism $\mathcal{A} \to \mathcal{E}$. The fibered product $\Psi = \varphi_1 \times_S \cdots \times_S \varphi_r$ of $\varphi_1, \ldots, \varphi_r \in \mathbf{Z}^g$ determines a proper morphism $\mathcal{A} \to \mathcal{E} \times_S \cdots \times_S \mathcal{E} = \mathcal{B}$ (r factors) over S. The restriction of $\mathcal{A} \to \mathcal{B}$ to a fiber above any $s \in S(\mathbf{C})$ induces a homomorphism of abelian varieties $\mathcal{A}_s \to \mathcal{B}_s$. We define the kernel $\ker \Psi$, as the fibered product of $\Psi : \mathcal{A} \to \mathcal{B}$ with the zero section $S \to \mathcal{B}$. We consider it as a closed subscheme of $\mathcal{A}$.

Lemma 2.5. (i) *Any flat subgroup scheme of $\mathcal{A}$ is equidimensional.*

(ii) *Let us assume that $\mathcal{A}$ is not isotrivial and let $H \subsetneq \mathcal{A}$ be a flat subgroup scheme. There exists $\varphi \in \mathbf{Z}^g \smallsetminus \{0\}$ such that $H \subset \ker \varphi$.*

(iii) *Let $\varphi_1, \ldots, \varphi_r \in \mathbf{Z}^g$ be $\mathbf{Z}$-linearly independent and $\Psi = \varphi_1 \times_S \cdots \times_S \varphi_r : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ as above. Then Ψ is smooth and $\ker \Psi$ is a non-singular flat subgroup scheme of $\mathcal{A}$ whose irreducible components have dimension $g - r + 1$.*

Proof. Let $H \subset \mathcal{A}$ be a flat subgroup scheme with irreducible components $H_1, \ldots, H_l$. There exists $s \in S(\mathbf{C})$ with the following property. For any $1 \le i \le l$ there is $P_i \in (H_i \smallsetminus \bigcup_{j \ne i} H_j)(\mathbf{C})$ with $\pi(P_i) = s$. On applying Exercise II 3.22 [9], which we call the Fiber Dimension Theorem from now on, we find $\dim_{P_i}(H_i)_s \ge \dim H_i - 1$. But equality must hold because $\pi|_{H_i} : H_i \to S$ is dominant. By choice of s, any irreducible component

of $(H_i)_s$ containing P_i is an irreducible component of H_s. But H_s is an algebraic group and therefore equidimensional. So $\dim H_s = \dim_{P_i}(H_i)_s = \dim H_i - 1$ is independent of i. Part (i) follows.

We turn to part (ii). Since $\mathcal{A}$ is not isotrivial, j is non-constant by Lemma 2.4 and hence dominant. The fiber H_s is an algebraic subgroup of $\mathcal{A}_s$. We fix $s \in S(\mathbf{C})$ with $j(s)$ transcendental. Since $H \neq \mathcal{A}$ we may also assume that $H_s \neq \mathcal{A}_s$. It is a classical fact that $\mathcal{E}_s$ does not have complex multiplication. The endomorphism ring of $\mathcal{A}_s = \mathcal{E}_s^g$ is $\mathrm{Mat}_g(\mathbf{Z})$. There exists $\varphi \in \mathbf{Z}^g \smallsetminus \{0\}$ with $H_s \subset \ker \varphi$. Let us consider $G = \varphi(H) \subset \mathcal{E}$. Since φ is a proper morphism, G is Zariski closed in $\mathcal{E}$. It is a subgroup scheme of $\mathcal{E}$. But $\mathcal{E} \to S$ is also proper, it follows that each irreducible component of G maps surjectively to S. So the fiber of $G \to S$ above s meets all irreducible components of G. On the other hand, it contains only the zero element of $\mathcal{E}_s$. The Fiber Dimension Theorem, implies that each irreducible component of G has dimension 1. Therefore, $G_{s'}$ is a finite group for all $s' \in S(\mathbf{C})$. The cardinality can even be bounded from above independently of s'. Hence after replacing φ by a positive integral multiple we may assume $G = \varphi(H)$ is the image of the zero section $S \to \mathcal{E}$. Hence $H \subset \ker \varphi$ and (ii) follows.

We now prove (iii). The restriction of Ψ to any fiber of $\mathcal{A} \to S$ induces a homomorphism between a g-th and an r-th power of an elliptic curve. Such a homomorphism is surjective because $\varphi_1, \ldots, \varphi_r$ are linearly independent and have as kernel an algebraic group of dimension $g - r$. These homomorphisms are smooth since domain and target are abelian varieties over a field of characteristic zero. Since $\mathcal{A} \to S$ is flat, Proposition 17.8.2 [7, EGA IV$_4$] implies that $\Psi : \mathcal{A} \to \mathcal{B}$ is smooth. Smoothness is preserved under base change, hence $\ker \Psi \to S$ is smooth. It follows that $\ker \Psi$ is a closed (possibly reducible) non-singular subvariety of $\mathcal{A}$. We see that $\ker \Psi$ is a subgroup scheme of $\mathcal{A}$. But $\ker \Psi \to S$ is flat and Proposition III 9.7 [9] implies that any irreducible component of $\ker \Psi$ dominates S. So $\ker \Psi$ is a flat subgroup scheme of $\mathcal{A}$. The statement on the dimension of $\ker \Psi$ follows from Corollary III 9.6 [9] and the fact that each fiber of $\ker \Psi \to S$ has dimension $g - r$. □

3. Torsion Points

3.1. **The Main Proposition.** The main result of this section is the following proposition. We count torsion points on subvarieties of the abelian scheme $\mathcal{A}_L$ from Section 2.1. In this section we consider both $\mathcal{A}_L$ and $Y(2)$ as defined over $\mathbf{C}$. The cardinality of a set M is denoted by $\#M$.

Proposition 3.1. *Let $X \subset \mathcal{A}_L$ be an irreducible closed subvariety defined over $\mathbf{C}$ which dominates $Y(2)$ with $\dim X = g$. Furthermore, we assume that*

(i) *either we have $\dim \varphi(X) \geq 2$ for every $\varphi \in \mathbf{Z}^g \smallsetminus \{0\}$,*
(ii) *or $\dim X = 1$ and X is not an irreducible component of a flat subgroup scheme of $\mathcal{E}$.*

Then there exist a (possible reducible) non-singular algebraic curve $C \subset \mathcal{A}_L$ and a constant $c = c(X) > 0$ such that if N is an integer with $N \geq c^{-1}$, then

$$\#\{P \in X(\mathbf{C});\ Q = [N](P) \in C(\mathbf{C}) \text{ and } \dim_Q [N](X) \cap C = 0\} \geq cN^{2g}. \tag{3.1}$$

We remark that in case (i) we have $\dim X \geq \dim \varphi(X) \geq 2$, so X cannot be a curve.

In the proof we will take C equal to $\ker[T]$ for some positive integer T. So the points P in (3.1) are torsion.

3.2. **Counting Torsion Points.** In this section we consider $\mathcal{A}_L(\mathbf{C})$ as a complex analytic space [6]; throughout the whole paper, all complex analytic spaces are assumed to be reduced. Then $\mathcal{A}_L(\mathbf{C})$ is even a complex manifold since $\mathcal{A}_L$ is non-singular variety. All references to a topology on $\mathcal{A}_L(\mathbf{C})$ will refer to the Euclidean topology unless stated otherwise with the exception that "irreducible" refers to the Zariski topology.

Recall that $\Sigma = \{z \in \mathbf{C};\ |z| < 1 \text{ and } |1-z| < 1\}$. The preimage $(\mathcal{A}_L)_\Sigma = {\pi_L}^{-1}(\Sigma) \subset \mathcal{A}_L(\mathbf{C})$ is an open complex submanifold of $\mathcal{A}_L(\mathbf{C})$. We also have a holomorphic (local) exponential map $\exp : \mathbf{C}^g \times \Sigma \to (\mathcal{A}_L)_\Sigma$. Its differential is an isomorphism at all points; to see this consider for example the Jacobian matrix.

Using the topological group $\mathbf{R}/\mathbf{Z}$ we now define a continuous function

$$\Xi : (\mathcal{A}_L)_\Sigma \to (\mathbf{R}/\mathbf{Z})^{2g}.$$

For any $P \in (\mathcal{A}_L)_\Sigma$ there is $w \in \mathbf{C}^g$ such that $\exp(w, \pi_L(P)) = P$ by Lemma 2.2. Because the columns of $\Omega(\pi_L(P))$ are an $\mathbf{R}$-basis of $\mathbf{C}^g$ there exists a unique $\xi \in \mathbf{R}^{2g}$ with $\Omega(\pi_L(P))\xi^{\mathsf{T}} = w$. We define $\Xi(P)$ to be the image of ξ in $(\mathbf{R}/\mathbf{Z})^{2g}$. We remark that if $w' \in \mathbf{C}^g$ with $\exp(w', \pi_L(P)) = P$, then the resulting ξ' will differ from ξ by an element in $\mathbf{Z}^{2g}$. So $\Xi(P)$ is well-defined. It remains to show that Ξ is continuous. Indeed, P has an open neighborhood U in $(\mathcal{A}_L)_\Sigma$ such that there exists a holomorphic map $\log : U \to \mathbf{C}^g \times Y(2)(\mathbf{C})$ with $\log(U)$ open in $\mathcal{A}_L(\mathbf{C})$ and with $\exp \circ \log$ the identity on U. Let $\log^0 : U \to \mathbf{C}^g$ denote log composed with the projection onto $\mathbf{C}^g$. The matrix

$$\begin{bmatrix} \Omega(\pi_L(P)) \\ \overline{\Omega}(\pi_L(P)) \end{bmatrix}$$

is invertible because the columns of $\Omega(\pi_L(P))$ are $\mathbf{R}$-linearly independent; the bar denotes complex conjugation. We set

$$\widetilde{\xi}(P) = \begin{bmatrix} \Omega(\pi_L(P)) \\ \overline{\Omega}(\pi_L(P)) \end{bmatrix}^{-1} \begin{bmatrix} \log^0(P) \\ \overline{\log^0(P)} \end{bmatrix} \in \mathbf{R}^{2g}.$$

Then $\widetilde{\xi}$ is clearly continuous on U. We let $\Xi|_U$ denote $\widetilde{\xi} : U \to \mathbf{R}^{2g}$ composed with the natural map $\mathbf{R}^{2g} \to (\mathbf{R}/\mathbf{Z})^{2g}$.

By Lemma 2.2(ii) the map $\Xi|_{(\mathcal{A}_L)_\lambda} : (\mathcal{A}_L)_\lambda(\mathbf{C}) \to (\mathbf{R}/\mathbf{Z})^{2g}$ is a group isomorphism for all $\lambda \in \Sigma$.

For any variety X defined over $\mathbf{C}$ let X^{ns} denote its Zariski open and dense subset of non-singular points. If $X \subset \mathcal{A}_L$ is a (possibly reducible) subvariety we set $X_\Sigma = (\mathcal{A}_L)_\Sigma \cap X(\mathbf{C})$.

Lemma 3.1. *Let X be an irreducible closed subvariety of $\mathcal{A}_L$ of dimension g. Let $P \in X_\Sigma^{\mathrm{ns}}$ such that $\Xi|_{X_\Sigma}^{-1}(\Xi(P))$ contains a countable neighborhood of P. Then P is isolated in $\Xi|_{X_\Sigma}^{-1}(\Xi(P))$ and $\Xi(X_\Sigma)$ contains a non-empty open subset of $(\mathbf{R}/\mathbf{Z})^{2g}$.*

Proof. Let $\log^0 : U \to \mathbf{C}^g$ and $\widetilde{\xi} : U \to \mathbf{R}^{2g}$ be as in the proof of continuity of Ξ where U is an open neighborhood of P. We may assume $X(\mathbf{C}) \cap U \subset X_\Sigma^{\mathrm{ns}}$; so, $X(\mathbf{C}) \cap U$ is a complex manifold of dimension g.

We define

$$Z = \{(Q, w) \in (X(\mathbf{C}) \cap U) \times \mathbf{C}^{2g};\ \Omega(\pi_L(Q))w - \log^0(Q) = 0\}. \tag{3.2}$$

Then Z, being the set of common zeros of holomorphic functions, is an analytic subset of the complex manifold $(X(\mathbf{C}) \cap U) \times \mathbf{C}^{2g}$. It contains (P, ξ) where $\xi = \widetilde{\xi}(P)$.

Using the Jacobian condition and the fact that the matrix $\Omega(\pi_L(Q))$ has rank g we conclude that Z is a complex submanifold of $(X(\mathbf{C}) \cap U) \times \mathbf{C}^{2g}$ of dimension $2g$.

Now let $q : Z \to \mathbf{C}^{2g}$ denote the projection onto the last $2g$ coordinates. One readily checks that

$$\Xi|_{X(\mathbf{C})\cap U}^{-1}(\Xi(P)) \times \{\xi\} = q^{-1}(\xi) = q^{-1}(q(P, \xi)).$$

By hypothesis, there is a countable neighborhood of (P, ξ) in $q^{-1}(q(P, \xi))$. But latter is a complex analytic subset of Z. So it contains (P, ξ) as an isolated point. Moreover, P is isolated in $\Xi|_{X_\Sigma}^{-1}(\Xi(P))$ and the first assertion follows.

We note that Z and $\mathbf{C}^{2g}$ are complex manifolds of equal dimension. The comment on page 64 [6] implies that q is a finite holomorphic map at (P, ξ). The proposition on page 107 in the same reference tells us that q is open at (P, ξ).

In particular, $q(Z)$ contains an open neighborhood $W' \subset \mathbf{C}^{2g}$ of ξ. Since $\xi \in \mathbf{R}^{2g}$ it follows that $W' \cap \mathbf{R}^{2g}$ is an open and non-empty subset of $\mathbf{R}^{2g}$. Finally, W, its image in $(\mathbf{R}/\mathbf{Z})^{2g}$, is open too.

By definition of Ξ and (3.2), any element of W is the image under Ξ of some element of $X(\mathbf{C}) \cap U \subset X_\Sigma$, so the second assertion follows. $\square$

We need a simple counting result on N-th roots of elements in $(\mathbf{R}/\mathbf{Z})^{2g}$.

Lemma 3.2. *Let W be a non-empty open subset of $(\mathbf{R}/\mathbf{Z})^{2g}$ then there exists a constant $c > 0$ with the following property. If $\xi_0 \in (\mathbf{R}/\mathbf{Z})^{2g}$ and if N is an integer with $N \geq c^{-1}$, then*

$$\#\{\xi \in W;\ N\xi = \xi_0\} \geq cN^{2g}.$$

Proof. There are $x_1, \ldots, x_{2g} \in \mathbf{R}$ and an $\epsilon \in (0, 1/2]$ such that the image of $U = \prod_{i=1}^{2g}(x_i - \epsilon, x_i + \epsilon)$ under the natural map $\mathbf{R}^{2g} \to (\mathbf{R}/\mathbf{Z})^{2g}$ is contained in W. Let $y = (y_1, \ldots, y_{2g}) \in \mathbf{R}^{2g}$ be a lift of ξ_0 and N an integer with $N \geq \epsilon^{-1}$. We have $NU - y = \prod_{i=1}^{2g}(Nx_i - \epsilon N - y_i, Nx_i + \epsilon N - y_i)$. Any real interval (a, b) contains at least $b - a - 1$ integers. Hence $\#(NU - y) \cap \mathbf{Z}^{2g} \geq (2\epsilon N - 1)^{2g} \geq (2\epsilon N - \epsilon N)^{2g} = \epsilon^{2g} N^{2g}$. If $Nu - y \in \mathbf{Z}^{2g}$ with $u \in U$, then $N\xi = \xi_0$ where $\xi \in W$ is the image of u in $(\mathbf{R}/\mathbf{Z})^{2g}$. If $u, u' \in U$ have equal image in $(\mathbf{R}/\mathbf{Z})^{2g}$, then $u = u'$ because $\epsilon \leq 1/2$. So $\#\{\xi \in W;\ N\xi = \xi_0\} \geq \epsilon^{2g} N^{2g}$ and the current lemma holds with $c = \epsilon^{2g}$. $\square$

The following remark on $\Xi : (\mathcal{A}_L)_\Sigma \to (\mathbf{R}/\mathbf{Z})^{2g}$ will be useful further down. By abuse of notation we also use Ξ to denote the continuous map $(\mathcal{E}_L)_\Sigma \to (\mathbf{R}/\mathbf{Z})^2$ if $g = 1$. For any $\varphi = (a_1, \ldots, a_g) \in \mathbf{Z}^g$ we have a commutative diagram

$$\begin{tikzcd} (\mathcal{A}_L)_\Sigma \arrow[r, "\Xi"] \arrow[d, "\varphi|_{(\mathcal{A}_L)_\Sigma}"'] & (\mathbf{R}/\mathbf{Z})^{2g} \arrow[d, "\Xi(\varphi)"] \\ (\mathcal{E}_L)_\Sigma \arrow[r, "\Xi"'] & (\mathbf{R}/\mathbf{Z})^2 \end{tikzcd} \tag{3.3}$$

where

$$\Xi(\varphi)(\xi_1,\ldots,\xi_{2g}) = (a_1\xi_1 + a_2\xi_3 + \cdots + a_g\xi_{2g-1}, a_1\xi_2 + a_2\xi_4 + \cdots + a_g\xi_{2g})$$

is a continuous homomorphism of groups.

By the next lemma, Proposition 3.1 holds for hypersurfaces which satisfy a non-degeneracy property with respect to Ξ.

Lemma 3.3. *Let X be an irreducible closed subvariety of $\mathcal{A}_L$ of dimension g. Let us assume that there exists $P \in X_\Sigma^{\mathrm{ns}}$ which is isolated in $\Xi|_{X_\Sigma}^{-1}(\Xi(P))$. Then Proposition 3.1 holds for X.*

Proof. The generic fiber $(\mathcal{A}_L)_\eta$ of $\mathcal{A}_L \to Y(2)$ is an abelian variety over the rational function field $\mathbf{C}(Y(2))$. It is the g-th power of $(\mathcal{E}_L)_\eta$, the generic fiber of $\mathcal{E}_L \to Y(2)$. Then $(\mathcal{E}_L)_\eta$ is an elliptic curve with non-constant j-invariant, cf. the left-hand side of (2.7). So $(\mathcal{E}_L)_\eta$ cannot have complex multiplication. Since $(\mathcal{A}_L)_\eta = (\mathcal{E}_L)_\eta^g$ we may identify the group of homomorphisms of algebraic groups $(\mathcal{A}_L)_\eta \to (\mathcal{E}_L)_\eta$ with $\mathbf{Z}^g$. Moreover, when replacing $\mathbf{C}(Y(2))$ by an algebraic closure K and regarding $(\mathcal{A}_L)_\eta$ and $(\mathcal{E}_L)_\eta$ over this larger field we do not get any new homomorphisms $(\mathcal{A}_L)_\eta \to (\mathcal{E}_L)_\eta$.

We recall that the Manin-Mumford Conjecture, a result of Raynaud [21], holds for an abelian variety defined over any field of characteristic 0. As a consequence there exist finitely many homomorphisms $\varphi_1,\ldots,\varphi_n \in \mathbf{Z}^g \smallsetminus \{0\}$ with the following property. Any torsion point of $(\mathcal{A}_L)_\eta(K)$ in $X_\eta(K)$ is contained in the kernel of some φ_i.

As usual, we consider each φ_i as a homomorphism $\mathcal{A}_L \to \mathcal{E}_L$. We may pick $\xi_0 \in (\mathbf{Q}/\mathbf{Z})^{2g} \subset (\mathbf{R}/\mathbf{Z})^{2g}$ which avoids $\ker \Xi(\varphi_1) \cup \cdots \cup \ker \Xi(\varphi_n)$. Then ξ_0 is torsion of order T, say. We shall apply Lemma 3.2 to W and ξ_0. But first we define C to be $\ker[T]$. This is just the T-torsion subgroup scheme of $\mathcal{A}_L$. It is a possibly reducible non-singular curve whose irreducible components dominate $Y(2)$ by Lemma 2.5(iii).

We note that P as in statement of this lemma satisfies the hypothesis of Lemma 3.1. So $\Xi(X_\Sigma)$ contains a non-empty open subset $W \subset (\mathbf{R}/\mathbf{Z})^{2g}$. Let $c > 0$ be as in Lemma 3.2 and $N \geq c^{-1}$. Recalling $W \subset \Xi(X_\Sigma)$, this lemma provides at least cN^{2g} points $P' \in X(\mathbf{C})$ such that $N\Xi(P') = \xi_0$. We proceed to show that any such P' lies in the set on the left of (3.1).

We have $0 = T\xi_0 = TN\Xi(P') = \Xi([TN]P')$ and because Ξ is fiberwise a group isomorphism this shows $Q \in C(\mathbf{C})$ for $Q = [N](P')$. We note $\Xi(Q) = \xi_0$. It remains to show

$$\dim_Q [N](X) \cap C = 0. \tag{3.4}$$

Let us assume the contrary and suppose that $Z' \subset [N](X) \cap C$ is an irreducible curve containing Q. Then Z' dominates $Y(2)$ since it is an irreducible component of C. We may choose an irreducible closed subvariety $Z \subset X$ with $[N](Z) = Z'$. Then $\dim Z = 1$ by the Fiber Dimension Theorem since $[N]$ has finite fibers. Let $P'' \in Z(\mathbf{C})$ with $[N](P'') = Q$. We have $[TN](Z) = [T](Z') \subset [T](C) = \epsilon_L(Y(2))$. Certainly, Z dominates $Y(2)$ so the function field of Z contains the function field of $Y(2)$. The former leads to a torsion point in $X_\eta(K) \subset (\mathcal{A}_L)_\eta(K)$. By our setup, this torsion point is in the kernel of some φ_i. This implies $Z \subset \ker \varphi_i$ and in particular $\varphi_i(P'') = 0$. Diagram (3.3) implies $\Xi(\varphi_i)(\Xi(P'')) = 0$. If we multiply this equality with N we get $0 = \Xi(\varphi_i)(\Xi([N](P''))) = \Xi(\varphi_i)(\Xi(Q)) = \Xi(\varphi_i)(\xi_0)$. But this contradicts our choice of ξ_0. So (3.4) must hold true. $\square$

3.3. **The Degenerate Case.** Throughout this section $X \subset \mathcal{A}_L$ will be an irreducible closed subvariety which dominates $Y(2)$. In general we pose no restriction on the dimension of X.

The previous discussion, in particular Lemma 3.3, suggests that we study the following property more careful. We call $P \in X_\Sigma^{\mathrm{ns}}$ degenerate if it is not isolated in $\Xi|_{X_\Sigma}^{-1}(\Xi(P))$. If all points of X_Σ^{ns} are degenerate, then we call X degenerate.

In order to handle the degenerate case we exploit monodromy of the family $\mathcal{A}_L \to Y(2)$ using the holomorphic map $\widetilde{\rho}_\xi$ from Section 2.3.

We say that $\xi = (\xi_1, \dots, \xi_{2g}) \in (\mathbf{R}/\mathbf{Z})^{2g}$ is in general position if $a_1\xi_1 + \cdots + a_{2g}\xi_{2g} \neq 0$ in $\mathbf{R}/\mathbf{Z}$ for all $(a_1, \dots, a_{2g}) \in \mathbf{Z}^{2g} \smallsetminus \{0\}$.

We define $\widetilde{\Sigma} = T(\Sigma) \subset \mathbf{H}$. This set is open in $\mathbf{H}$ because T is holomorphic and non-constant. We remark that Lemma 2.1 implies $\Lambda(\widetilde{\Sigma}) = \Sigma$.

The next lemma uses Kronecker's Theorem in diophantine approximation to extract information on the image of $\widetilde{\rho}_\xi$ for ξ in general position. This image is what is refered to colloquially as an analytic subgroup scheme in the introduction.

Lemma 3.4. *Let $\xi = (\xi_1, \dots, \xi_{2g}) \in (\mathbf{R}/\mathbf{Z})^{2g}$ such that $(\xi_2, \xi_4, \dots, \xi_{2g}) \in (\mathbf{R}/\mathbf{Z})^g$ is in general position. Then $\widetilde{\rho}_\xi(\mathbf{H})$ is Zariski dense in $\mathcal{A}_L$.*

Proof. Say Z is the Zariski closure of $\widetilde{\rho}_\xi(\mathbf{H})$ in $\mathcal{A}_L$.

We fix $\tau \in \widetilde{\Sigma} \subset \mathbf{H}$; then $\tau = T(\lambda)$ for some $\lambda \in \Sigma$. By hypothesis we have $\widetilde{\rho}_\xi(\tau + 2k) \in Z(\mathbf{C})$ for all $k \in \mathbf{Z}$. We apply Lemma 2.3(ii) by induction to obtain

$$\widetilde{\rho}_\xi(\tau + 2k) = \widetilde{\rho}_{\xi + 2k(\xi_2, 0, \dots, \xi_{2g}, 0)}(\tau) \in Z(\mathbf{C}).$$

Kronecker's Theorem IV page 53 [4] and our hypothesis on ξ imply that

$$\{(2k\xi_2, 2k\xi_4, \dots, 2k\xi_{2g});\ k \in \mathbf{Z}\} \quad \text{lies dense in} \quad (\mathbf{R}/\mathbf{Z})^g.$$

Now

$$f(z_1, \dots, z_g) = \exp(\Omega(\lambda)(z_1, \xi_2, z_2, \xi_4, \dots, z_g, \xi_{2g})^{\mathrm{T}}, \lambda)$$

is a $\mathbf{Z}^g$-periodic holomorphic map $f : \mathbf{C}^g \to \mathcal{A}_L(\mathbf{C})$. We recall Lemma 2.3(i) to deduce that f takes values in $Z(\mathbf{C})$ at $(2k\xi_2, 2k\xi_4, \dots, 2k\xi_{2g})$ for all $k \in \mathbf{Z}$. By continuity we conclude $f(\mathbf{R}^g) \subset Z(\mathbf{C})$. So $f^{-1}(Z(\mathbf{C}))$ is a complex analytic subset of $\mathbf{C}^g$ which contains $\mathbf{R}^g$. But the only such set is $\mathbf{C}^g$ itself. Hence $f(\mathbf{C}^g) \subset Z(\mathbf{C})$. From the definition (2.9) of Ω and because ω_1 never vanishes we see $f(\mathbf{C}^g) = (\mathcal{A}_L)_\lambda(\mathbf{C}) \subset Z(\mathbf{C})$.

We let τ vary over $\widetilde{\Sigma}$ and use $\Lambda(\widetilde{\Sigma}) = \Sigma$ to find $(\mathcal{A}_L)_\Sigma \subset Z(\mathbf{C})$. Now $(\mathcal{A}_L)_\Sigma$ is non-empty and open in $\mathcal{A}_L(\mathbf{C})$ with respect to the Euclidean topology. It is therefore Zariski dense in $\mathcal{A}_L$ and thus $Z = \mathcal{A}_L$. $\square$

Without much effort one can strengthened this argument to show that $\widetilde{\rho}_\xi(\mathbf{H})$ is not contained in a proper analytic subset of $\mathcal{A}_L(\mathbf{C})$.

The next lemma uses the fact that $Y(2)$ has dimension 1 in an essential way.

Lemma 3.5. *If $P \in X_\Sigma^{\mathrm{ns}}$ is degenerate, then $\widetilde{\rho}_{\Xi(P)}(\mathbf{H}) \subset X(\mathbf{C})$.*

Proof. For brevity we set $\xi = \Xi(P)$. By the first assertion of Lemma 3.1 the fiber $Z = \Xi|_{X_\Sigma}^{-1}(\xi)$ is uncountable.

We claim that $\Xi^{-1}(\xi) \subset \widetilde{\rho}_\xi(\widetilde{\Sigma})$. Indeed, say $P' \in (\mathcal{A}_L)_\Sigma$ with $\Xi(P') = \xi$. By Lemma 2.1(i) we have $\Lambda(\tau) = \pi_L(P')$ with $\tau = T(\pi_L(P'))$. Hence $P' = \exp(\Omega(\pi_L(P'))\xi, \pi_L(P')) = \widetilde{\rho}_\xi(\tau)$ by Lemma 2.3(i), as desired.

The claim implies $Z \subset \widetilde{\rho}_\xi(\widetilde{\Sigma})$. So $Y = \widetilde{\rho}_\xi^{-1}(X(\mathbf{C}))$ is an uncountable complex analytic subset of $\mathbf{H}$. In particular, there is $\tau \in Y$ with $\dim_\tau Y \geq 1$.

Now we make use of the trivial, but crucial, fact that $\mathbf{H}$ has dimension 1 at all points. So we must have $\dim_\tau Y = \dim_\tau \mathbf{H}$. With this, the Identity Lemma, page 167 [6] implies $Y = \mathbf{H}$. In other words, $\widetilde{\rho}_\xi(\mathbf{H}) \subset X(\mathbf{C})$. □

The following possibly well-known statement helps to study the degenerate case.

Lemma 3.6. *Let A be an abelian variety of dimension g and $Y \subset A$ an irreducible closed subvariety. We let $s : A^g \to A$ denote the morphism $(P_1, \ldots, P_g) \to P_1 + \cdots + P_g$. Then $s(Y^g)$ is the translate of an abelian subvariety of A.*

Proof. After translating Y we may assume $0 \in Y(\mathbf{C})$. We also immediately reduce to the case $Y \neq 0$ and $s(Y^g) \neq A$. For $1 \leq k \leq g$ we let $s_k : A^k \to A$ denote the morphism $(P_1, \ldots, P_k) \to P_1 + \cdots + P_k$. Then $s_1(Y) \subset s_2(Y^2) \subset \cdots \subset s_g(Y^g)$ because $0 \in Y(\mathbf{C})$. Each $s_k(Y^k)$ is an irreducible closed subvariety of A. The dimensions satisfy $1 \leq \dim s_1(Y) \leq \cdots \leq \dim s_g(Y^g) \leq g-1$. By the Pigeonhole Principle there exist $k < l$ such that $\dim s_k(Y^k) = \dim s_l(Y^l)$, we may assume $l = k+1$. We must even have $s_k(Y^k) = s_{k+1}(Y^{k+1}) = B$ because these varieties are irreducible. Certainly, $0 \in B(\mathbf{C})$ and if $P_i, Q_i \in Y(\mathbf{C})$ for $1 \leq i \leq k$, then $(P_1 + \cdots + P_k + Q_1) + Q_2 + \cdots + Q_k \in B(\mathbf{C}) + Q_2 + \cdots + Q_k$. By induction we get $P_1 + \cdots + Q_k \in B(\mathbf{C})$, so B is closed under addition. If $P \in B(\mathbf{C})$ is fixed, then $Q \mapsto Q + P$ defines a proper morphism $B \to B$ with finite fibers. Comparing dimension we see that this morphism is surjective, so there is $Q \in B(\mathbf{C})$ with $P + Q = 0$. Hence B is closed under inversion too. Therefore, B is an abelian subvariety of A. It follows that $s(Y^g) = B$. □

Lemma 3.7. *Let us assume that X is degenerate and $X \neq \mathcal{A}_L$. There exist $\varphi \in \mathbf{Z}^g \smallsetminus \{0\}$ and an uncountable subset $\Sigma' \subset \Sigma$ such that for any $\lambda \in \Sigma'$ there is an irreducible component X'_λ of X_λ with* $\dim \varphi(X'_\lambda) = 0$.

Proof. Lemmas 3.4 and 3.5 imply that if $P \in X_\Sigma^{\mathrm{ns}}$, then $\Xi(P)$ is not in general position. If $a = (a_1, \ldots, a_{2g}) \in \mathbf{Z}^{2g}$ we let G_a denote the closed subgroup $\{(\xi_1, \ldots, \xi_{2g}) \in (\mathbf{R}/\mathbf{Z})^{2g};\ a_1\xi_1 + \cdots + a_{2g}\xi_{2g} = 0\} \subset (\mathbf{R}/\mathbf{Z})^{2g}$. Then

$$X_\Sigma = (X \smallsetminus X^{\mathrm{ns}})_\Sigma \cup \bigcup_{a \in \mathbf{Z}^{2g} \smallsetminus \{0\}} \Xi^{-1}(G_a) \cap X_\Sigma.$$

Each of the countably many sets in the union above is closed in X_Σ. Of course, X_Σ is non-empty because X dominates $Y(2)$. So the interior in X_Σ of one of the sets

$$(X \smallsetminus X^{\mathrm{ns}})_\Sigma, \quad \Xi^{-1}(G_a) \cap X_\Sigma \text{ with } a \in \mathbf{Z}^{2g} \smallsetminus \{0\}$$

is non-empty by the Baire Category Theorem. But it cannot be $(X \smallsetminus X^{\mathrm{ns}})_\Sigma$ since the singular locus is a Zariski closed and proper subset of X. So there is $a \in \mathbf{Z}^{2g} \smallsetminus \{0\}$ and a non-empty open set $U \subset X_\Sigma$ with

$$\Xi(U) \subset G_a. \tag{3.5}$$

We remark that $\pi_L|_{X(\mathbf{C})} : X(\mathbf{C}) \to Y(2)(\mathbf{C})$ is a non-constant holomorphic function and $Y(2)(\mathbf{C}) \subset \mathbf{C}$ is open. So $\pi_L(U)$ is open in $Y(2)(\mathbf{C})$ by the corollary on page 109 [6]. Let Σ' be the set of transcendental elements in $\pi_L(U)$. Then Σ' is uncountable and

if $\lambda \in \Sigma'$ then $(\mathcal{E}_L)_\lambda$ does not have complex multiplication because its j-invariant is the transcendental number (2.7).

Let $V \subset \mathbf{R}^{2g}$ be the preimage of G_a under $\mathbf{R}^{2g} \to (\mathbf{R}/\mathbf{Z})^{2g}$. Say $\lambda \in \Sigma'$ and let X'_λ be an irreducible component of the fiber X_λ with $U_\lambda = X'_\lambda(\mathbf{C}) \cap U \neq \emptyset$. It follows from (3.5) that $U_\lambda \subset \exp(\Omega(\lambda)V, \lambda)$. In fact, we even have $P_1 + \cdots + P_g \in \exp(\Omega(\lambda)V, \lambda)$ for $P_1, \ldots, P_g \in U_\lambda$ since V is a group. In the notation of Lemma 3.6 we can restate this as

$$s(U_\lambda^g) \subset \exp(\Omega(\lambda)V, \lambda). \tag{3.6}$$

We claim that $s(X'^g_\lambda) \neq (\mathcal{A}_L)_\lambda$. This will complete the proof of the lemma in view of Lemma 3.6 and the following argument. Any proper algebraic subgroup of $(\mathcal{A}_L)_\lambda$ is in the kernel of a non-trivial homomorphism $(\mathcal{A}_L)_\lambda \to (\mathcal{E}_L)_\lambda$ which can be identified with an element $\varphi_\lambda \in \mathbf{Z}^g \smallsetminus \{0\}$ since $(\mathcal{E}_L)_\lambda$ lacks complex multiplication. Moreover, since $\mathbf{Z}^g \smallsetminus \{0\}$ is countable we may assume that φ_λ is independent of λ after replacing Σ' by an uncountable subset.

To prove our claim let us suppose $s(X'^g_\lambda) = (\mathcal{A}_L)_\lambda$ and derive contradiction. The set U^g_λ is non-empty and open in $X'^g_\lambda(\mathbf{C})$, so it lies Zariski dense. Hence it contains a point where the variety X'^g_λ is non-singular and where $s|_{X'^g_\lambda}$ has maximal rank. The holomorphic map $s|_{X'^g_\lambda(\mathbf{C})}$ is open at this point. Thus $s(U^g_\lambda)$ contains a non-empty open subset of $(\mathcal{A}_L)_\lambda(\mathbf{C})$. So (3.6) implies that V contains a non-empty open subset of $\mathbf{R}^{2g}$, a contradiction. □

The statement of this lemma is void if $\dim X = 1$. Indeed, in this case every irreducible component of X_λ is a point and, as such, the translate of the trivial abelian subvariety of $(\mathcal{A}_L)_\lambda$.

Lemma 3.8. *Let us assume that X is degenerate and $X \neq \mathcal{A}_L$. There exists $\varphi \in \mathbf{Z}^g \smallsetminus \{0\}$ such that* $\dim \varphi(X) \leq 1$.

Proof. Let φ, Σ', and X'_λ be as in Lemma 3.7. The Fiber Dimension Theorem applied to $\pi_L|_X : X \to Y(2)$ implies $X'_\lambda \geq \dim X - 1$ for all $\lambda \in \Sigma'$. So, the fiber of $\varphi|_X : X \to \mathcal{E}_L$ through any point of $\bigcup_{\lambda \in \Sigma'} X'_\lambda(\mathbf{C})$ has dimension at least $\dim X - 1$. By comparing dimensions we see that $\bigcup_{\lambda\in\Sigma'} X'_\lambda(\mathbf{C})$ is Zariski dense in X. We again apply the Fiber Dimension Theorem to conclude that there is $P \in \bigcup_{\lambda\in\Sigma'} X'_\lambda(\mathbf{C})$ such that $\dim_P \varphi|_X^{-1}(\varphi(P)) = \dim X - \dim \varphi(X)$. So $\dim \varphi(X) \leq 1$, as desired. □

3.4. **The Case of Curves.** We first handle case (ii) of Proposition 3.1. The case of curves will be the starting point of an inductive argument eventually leading to the proof of Theorem 1.3.

Proof of Proposition 3.1(ii). Let $P \in X^{\mathrm{ns}}_\Sigma$ and $\lambda = \pi_L(P)$. In the current case $X \subset \mathcal{E}_L$ is a curve which dominates $Y(2)$. So $X_\lambda(\mathbf{C})$ is finite of cardinality bounded independently of P. By Lemma 3.3 we may assume that P is not isolated in $\Xi|_{X_\Sigma}^{-1}(\Xi(P))$. So Lemma 3.5 implies $\widetilde{\rho}_\xi(\mathbf{H}) \subset X(\mathbf{C})$. We set $\xi = (\xi_1, \xi_2) = \Xi(P) \in (\mathbf{R}/\mathbf{Z})^2$.

First we use local monodromy around 0 by applying Lemma 2.3(ii) to see

$$\widetilde{\rho}_{\xi+2k(\xi_2,0)}(T(\lambda)) = \widetilde{\rho}_\xi(T(\lambda)+2k) \in X_\lambda(\mathbf{C}) \quad \text{for all} \quad k \in \mathbf{Z}.$$

By Lemma 2.3(i) and the Pigeonhole Principle there is an integer $N \geq 1$ independent of P with $N\xi_2 = 0$.

To handle ξ_1 we need local monodromy around the cusp 1. We use Lemma 2.3(iii) and obtain

$$\widetilde{\rho}_{\xi-4k(0,\xi_1)}(T(\lambda)) \in X_\lambda(\mathbf{C}) \quad \text{for all} \quad k \in \mathbf{Z}.$$

As before we have $N\xi_1 = 0$, after possibly adjusting N.

Because N is independent of P we obtain

$$\widetilde{\rho}_\xi(T(\pi_L(P))) \in X \cap \ker[N] \quad \text{for all} \quad P \in X_\Sigma^{\mathrm{ns}}.$$

Therefore, $X \cap \ker[N]$ is infinite and so $X \subset \ker[N]$. This is a contradiction since $\ker[N]$ is one-dimensional flat subgroup scheme of $\mathcal{E}$ by Lemma 2.5(iii). □

3.5. **Proof of the Proposition.** Let X be as in the statement of the proposition. Part (ii) was already proved in Section 3.4 and it remains to show (i). The hypothesis and Lemma 3.8 imply that X is not degenerate. The proof now follows from Lemma 3.3. □

4. Intersection Numbers

In this section we use Proposition 3.1 with a theorem of Siu to construct an auxiliary non-zero global section of a certain line bundle. We then deduce the following height inequality.

Proposition 4.1. *Let $X \subset \mathcal{A}_L$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$ which dominates $Y(2)$, has dimension g, and satisfies (i) or (ii) of Proposition 3.1. There exists a constant $c > 0$ with the following property. For any integer $N \geq c^{-1}$ there is a non-empty Zariski open subset $U \subset X$ and a constant $c'(N)$ such that*

$$h_{\mathcal{A}_L}([2^N](P)) \geq c4^N h_{\mathcal{A}_L}(P) - c'(N)$$

for all $P \in U(\overline{\mathbf{Q}})$.

4.1. **Degree and Height Lower Bounds.** The proof Proposition 4.1 is based on a degree estimate.

Let f be a rational map between two irreducible varieties. Then $\mathrm{dom}(f)$ denotes the domain of f. If source and target of f have equal dimension we define $\deg f$, the degree of f, as follows. If f is dominant, then $\deg f$ is the degree of the (finite) extension of function fields induced by f. If f is not dominant, we set $\deg f = 0$.

Lemma 4.1. *Let X be an irreducible variety defined over $\mathbf{C}$ of dimension g and let $f : X \dashrightarrow \mathbf{P}^g$ be a rational map. If $Q \in \mathbf{P}^g(\mathbf{C})$, then the number of zero-dimensional irreducible components of $f^{-1}(Q)$ is at most* $\deg f$.

Proof. In this proof, any mention to a topology on $X(\mathbf{C})$ or $\mathbf{P}^g(\mathbf{C})$ refers to the Euclidean topology if not stated otherwise.

Let $P_1, \ldots, P_d \in \mathrm{dom}(f)(\mathbf{C})$ be distinct and isolated in the fiber of f above Q with respect to the Zariski topology. The P_i are also isolated with respect to the Euclidean topology. We may assume $d \geq 1$.

We regard $\mathrm{dom}(f)(\mathbf{C})$ and $\mathbf{P}^g(\mathbf{C})$ as g-dimensional complex analytic spaces and f as a holomorphic map between them. The comment on page 64 [6] implies that f is a finite holomorphic map at P_i for $1 \leq i \leq d$. By the proposition on page 107 [6] we conclude that f is an open map at each P_i. Hence there exists an open neighborhood U_i of P_i in $\mathrm{dom}(f)(\mathbf{C})$ such that $f|_{U_i}$ is an open mapping. We may assume that the U_i

are pairwise disjoint. The intersection $W = \bigcap_{i=1}^d f(U_i)$ is open in $\mathbf{P}^g(\mathbf{C})$ and contains Q. Let $Q' \in W$. There exists $P_i' \in U_i$ with $f(P_i') = Q'$. The resulting P_i' are pairwise distinct, so the fiber of f above any point in W has cardinality at least d.

There exists a Zariski closed and proper $Z \subset \mathbf{P}^g$ such that $\# f^{-1}(Q') = \deg f$ for all $Q' \in (\mathbf{P}^g \setminus Z)(\mathbf{C})$. But W, being a non-empty open subset of $\mathbf{P}^g(\mathbf{C})$, is Zariski dense in $\mathbf{P}^g$ and hence must meet $(\mathbf{P}^g \setminus Z)(\mathbf{C})$. We obtain $d \le \deg f$. □

The previous lemma can fail with $\mathbf{P}^g$ replaced by a (non-normal) variety. Indeed, the normalization morphism of a curve with a node has degree 1 but more than one point above the node.

Let $\mathcal{O}(1)$ denote the unique ample generator of the Picard group of projective space. For an irreducible closed subvariety X of projective space we let $\deg(X)$ be its geometric degree $(\mathcal{O}(1)^{\cdot \dim X}.[X])$; we refer to Chapters 1 and 2 [5] for a treatment of the intersection theory needed here.

We come to a preliminary height lower bound which depends on a Theorem of Siu.

Lemma 4.2. *Let $X \subset \mathbf{P}^n$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$ of dimension $g \ge 1$. Let $f : X \dashrightarrow \mathbf{P}^g$ be the rational map given by $f = [F_0 : \cdots : F_g]$ where F_i are homogeneous polynomials that are not all identically zero on X and have equal degree at most $D \ge 1$. There is a constant $c = c(X, f)$ and a proper and Zariski closed subset $Z \subset X$ such that $F_0, \ldots, F_g$ have no common zeros on $X \setminus Z$ and*

$$h(f(P)) \ge \frac{1}{4^g \deg(X)} \frac{\deg f}{D^{g-1}} h(P) - c$$

for all $P \in (X/Z)(\overline{\mathbf{Q}})$.

Proof. Without loss of generality we may assume $\deg f \ge 1$.

The rational map f need not be a morphism of varieties. In order to resolve the points of indeterminacy we define the Zariski closure of its graph

$$\Gamma = \overline{\{(P, f(P));\ P \in \mathrm{dom}(f)(\overline{\mathbf{Q}})\}} \subset X \times \mathbf{P}^g.$$

This is an irreducible projective variety with $\dim \Gamma = \dim X = g$. The morphism $\mathrm{dom}(f) \hookrightarrow \Gamma$ determined by $P \mapsto (P, f(P))$ is birational. Say $\pi_1 : \Gamma \to X$ and $\pi_2 : \Gamma \to \mathbf{P}^g$ are the two projection morphisms. Then $\pi_2(\Gamma) = \mathbf{P}^g$ by the Fiber Dimension Theorem and since $\pi_2|_\Gamma$ has finite fibers generically.

From functorial properties of the height we see that

$$h_{\Gamma, \pi_1^* \mathcal{O}(1)|_X}(P, P') = h(P) \quad \text{and} \quad h_{\Gamma, \pi_2^* \mathcal{O}(1)}(P, P') = h(P') \quad \text{for} \quad (P, P') \in \Gamma(\overline{\mathbf{Q}})$$

are valid choices for height functions; we recall that h is the projective height. In order to prove this lemma it is enough to show that there exists $c \in \mathbf{R}$ with

$$h_{\Gamma, \pi_2^* \mathcal{O}(1)}(Q) \ge \frac{1}{4^g \deg(X)} \frac{\deg f}{D^{g-1}} h_{\Gamma, \pi_1^* \mathcal{O}(1)|_X}(Q) - c,$$

for all Q in a Zariski open dense subset of Γ. Using functorial properties of the height, this inequality holds if some positive integral power of the line bundle

$$\pi_2^* \mathcal{O}(1)^{\otimes 4^g \deg(X) D^{g-1}} \otimes \pi_1^* \mathcal{O}(1)|_X^{\otimes(-\deg f)}$$

admits a non-zero global section. By a result of Siu, Theorem 2.2.15 [13], such a section exists provided we have the following inequality on intersection numbers

$$\left((\pi_2^*\mathcal{O}(1)^{\otimes 4^g \deg(X) D^{g-1}})^{\cdot g}.[\Gamma]\right) \overset{?}{>} g\left(\pi_1^*\mathcal{O}(1)|_X^{\otimes \deg f}.(\pi_2^*\mathcal{O}(1)^{\otimes 4^g \deg(X) D^{g-1}})^{\cdot(g-1)}.[\Gamma]\right).$$

By multilinearity of intersection numbers the left-hand side is $4^{g^2}\deg(X)^g D^{g(g-1)}(\pi_2^*\mathcal{O}(1)^{\cdot g}.[\Gamma])$ while the right-hand side is $4^{g(g-1)}g\deg(X)^{g-1}D^{(g-1)^2}(\deg f)(\pi_1^*\mathcal{O}(1)|_X.\pi_2^*\mathcal{O}(1)^{\cdot(g-1)}.[\Gamma])$. Hence our lemma follows if we can prove

$$4^g\deg(X)D^{g-1}\,(\pi_2^*\mathcal{O}(1)^{\cdot g}.[\Gamma]) \overset{?}{>} g(\deg f)\left(\pi_1^*\mathcal{O}(1)|_X.\pi_2^*\mathcal{O}(1)^{\cdot(g-1)}.[\Gamma]\right). \tag{4.1}$$

We proceed by proving this inequality. The projection formula implies

$$(\pi_2^*\mathcal{O}(1)^{\cdot g}.[\Gamma]) = (\deg\pi_2)(\mathcal{O}(1)^{\cdot g}.[\pi_2(\Gamma)]) = (\deg\pi_2)(\mathcal{O}(1)^{\cdot g}.[\mathbf{P}^g]) = \deg\pi_2.$$

The birational morphism $\mathrm{dom}(f)\hookrightarrow\Gamma$ composed with π_2 is nothing other then $f:\mathrm{dom}(f)\to\mathbf{P}^g$. Hence we have $\deg\pi_2=\deg f$ and so

$$(\pi_2^*\mathcal{O}(1)^{\cdot g}.[\Gamma]) = \deg f.$$

By (4.1) it suffices to show

$$4^g\deg(X)D^{g-1} \overset{?}{>} g\left(\pi_1^*\mathcal{O}(1)|_X.\pi_2^*\mathcal{O}(1)^{\cdot(g-1)}.[\Gamma]\right). \tag{4.2}$$

Let $\rho_{1,2}$ denote the projections of $\mathbf{P}^n\times\mathbf{P}^g$ onto the first and second factor, respectively. If $Z\subset\mathbf{P}^n\times\mathbf{P}^g$ is an irreducible closed subvariety we set

$$H_Z = \sum_{\substack{i+j=\dim Z\\ i,j\geq 0}}\binom{\dim Z}{i}\left(\rho_1^*\mathcal{O}(1)^{\cdot i}.\rho_2^*\mathcal{O}(1)^{\cdot j}.[Z]\right)U^iV^j\in\mathbf{Z}[U,V].$$

This is the highest homogeneous part of the biprojective Hilbert polynomial of Z multiplied by $(\dim Z)!$, cf. [16]. It is homogeneous of degree $\dim Z$ with non-negative integer coefficients. In particular, $H_Z(D,1)\geq 0$.

Our projective variety Γ is an irreducible component of the intersection of $X\times\mathbf{P}^g$ with the set of common zeros of

$$F_i(X_0,\ldots,X_n)-Y_i\in\overline{\mathbf{Q}}[X_0,\ldots,X_n,Y_0,\ldots,Y_g]\quad(0\leq i\leq g); \tag{4.3}$$

here X_i,Y_i are projective coordinates on $\mathbf{P}^n$ and $\mathbf{P}^g$, respectively. These polynomials are bihomogeneous of bidegree $(\deg F_i,1)$. We recall $\deg F_i\leq D$. Philippon's Proposition 3.3 [16] implies $\sum_{\Gamma'}H_{\Gamma'}(D,1)\leq H_{X\times\mathbf{P}^g}(D,1)$ where the sum runs over all irreducible components Γ' cut out on $X\times\mathbf{P}^g$ by the polynomials (4.3). For any Γ' we have $H_{\Gamma'}(D,1)\geq 0$. By forgetting about all irreducible components except Γ we see

$$H_\Gamma(D,1)\leq H_{X\times\mathbf{P}^g}(D,1). \tag{4.4}$$

Now $\binom{g}{1}\left(\pi_1^*\mathcal{O}(1)|_X.\pi_2^*\mathcal{O}(1)^{\cdot(g-1)}.[\Gamma]\right)D$ is one term in the sum $H_\Gamma(D,1)$. Since all other terms are non-negative, (4.4) gives

$$g\left(\pi_1^*\mathcal{O}(1)|_X.\pi_2^*\mathcal{O}(1)^{\cdot(g-1)}.[\Gamma]\right)D\leq H_{X\times\mathbf{P}^g}(D,1). \tag{4.5}$$

To complete the proof of (4.2) we now bound $H_{X\times\mathbf{P}^g}(D,1)$ from above. We have $\dim X\times\mathbf{P}^g=2g$, so by definition

$$H_{X\times\mathbf{P}^g}(D,1)=\sum_{i=0}^{2g}\binom{2g}{i}\left(\rho_1^*\mathcal{O}(1)^{\cdot i}.\rho_2^*\mathcal{O}(1)^{\cdot(2g-i)}.[X\times\mathbf{P}^g]\right)D^i. \tag{4.6}$$

Two applications of the projection formula lead to

$$\begin{aligned}(\rho_1^*\mathcal{O}(1)^{\cdot i}.\rho_2^*\mathcal{O}(1)^{\cdot(2g-i)}.[X\times\mathbf{P}^g]) &= \left(\mathcal{O}(1)^{\cdot i}.\rho_{1*}(\rho_2^*\mathcal{O}(1)^{\cdot(2g-i)}.[X\times\mathbf{P}^g])\right)\\ &= \left(\mathcal{O}(1)^{\cdot(2g-i)}.\rho_{2*}(\rho_1^*\mathcal{O}(1)^{\cdot i}.[X\times\mathbf{P}^g])\right).\end{aligned} \tag{4.7}$$

The cycle class $\rho_2^*\mathcal{O}(1)^{\cdot(2g-i)}.[X\times\mathbf{P}^g]$ on $\mathbf{P}^n\times\mathbf{P}^g$ is trivial if $2g-i>g$ and $\rho_1^*\mathcal{O}(1)^{\cdot i}.[X\times\mathbf{P}^g]$ is trivial if $i>\dim X=g$. Therefore, all terms in (4.6) with $i\neq g$ vanish. We are left with

$$H_{X\times\mathbf{P}^g}(D,1)=\binom{2g}{g}(\rho_1^*\mathcal{O}(1)^{\cdot g}.\rho_2^*\mathcal{O}(1)^{\cdot g}.[X\times\mathbf{P}^g])\,D^g.$$

We find $(\rho_1^*\mathcal{O}(1)^{\cdot g}.\rho_2^*\mathcal{O}(1)^{\cdot g}.[X\times\mathbf{P}^g])=\deg(X)$ on inserting $i=g$ in (4.7). We recall (4.5) and conclude

$$(\pi_1^*\mathcal{O}(1)|_X.\pi_2^*\mathcal{O}(1)^{\cdot(g-1)}.[\Gamma])\leq\frac{1}{g}\binom{2g}{g}\deg(X)D^{g-1}.$$

So inequality (4.2) holds true since $\binom{2g}{g}<4^g$. As stated above, this completes the proof. □

Before we come to the proof of Proposition 4.1 we give an explicit formula for the duplication morphism on $\mathcal{E}_L$.

Lemma 4.3. *Let $N\in\mathbf{N}$, there exist polynomials $G_{N,0},G_{N,1},G_{N,2}\in\mathbf{Z}[X_0,X_1,X_2,X_3]$ of total degree at most $2\cdot4^N$ and homogeneous of degree 4^N in X_0,X_1,X_2 with the following properties. If $([x:y:z],\lambda)\in\mathcal{E}_L(\mathbf{C})$ then $G_{N,i}(x,y,z,\lambda)\neq0$ for some $i\in\{0,1,2\}$ and*

$$[2^N]([x:y:z],\lambda)=([G_{N,0}(x,y,z,\lambda):G_{N,1}(x,y,z,\lambda):G_{N,2}(x,y,z,\lambda)],\lambda).$$

Proof. Let

$$\begin{aligned}G_{1,0}&=2X_1X_2^3X_3^2+(2X_0^3X_1-6X_0^2X_1X_2)X_3+(2X_0^3X_1+2X_0X_1^3),\\ G_{1,1}&=(-4X_0^2X_2^2+6X_0X_2^3-X_2^4)X_3^3+(-X_0^4+9X_0^3X_2-17X_0^2X_2^2+6X_0X_2^3-4X_1^2X_2^2)X_3^2\\ &\quad+(-2X_0^4+9X_0^3X_2-4X_0^2X_2^2+3X_0X_1^2X_2-4X_1^2X_2^2)X_3+(-X_0^4+X_1^4),\\ G_{1,2}&=8X_1^3X_2.\end{aligned}$$

These three polynomials have no common zeros on $\mathcal{E}_L\subset\mathbf{P}^2\times Y(2)$. They are homogeneous of degree 4 in X_0,X_1,X_2 and of degree at most 3 in X_3. If $([x:y:z],\lambda)\in\mathcal{E}_L(\mathbf{C})$, the duplication formula on page 59 [23] implies

$$[2]([x:y:z],\lambda)=([G_{1,0}(x,y,z,\lambda):G_{1,1}(x,y,z,\lambda):G_{1,2}(x,y,z,\lambda)],\lambda).$$

We define $G_{Ni}=G_{1,i}(G_{N-1,0},G_{N-1,1},G_{N-1,2},X_3)$ inductively. These polynomials describe $[2^N]$ since $G_{N,0},G_{N,1},G_{N,2}\in\mathbf{Z}[X_0,X_1,X_2,X_3]$ have no common zero on $\mathcal{E}_L$. By induction we find that G_{Ni} are homogeneous of degree 4^N in X_0,X_1,X_2 and of degree at most 4^N-1 in X_3. So their total degree is at most $2\cdot4^N-1$. □

Proof of Proposition 4.1. Let $X \subset \mathcal{A}_L$ be as in the hypothesis. Recall that $\mathcal{A}_L \subset (\mathbf{P}^2)^g \times \mathbf{P}^1$ is quasi-projective. The Segre embedding $(\mathbf{P}^2)^g \times \mathbf{P}^1 \hookrightarrow \mathbf{P}^n$, with $n = 2 \cdot 3^g - 1$, enables us to embed $\mathcal{A}_L$ into projective space. Under this embedding, $\mathcal{A}_L$ becomes Zariski open in its Zariski closure. By abuse of notation we will suppose $\mathcal{A}_L \subset \mathbf{P}^n$. If $P \in \mathcal{A}_L(\overline{\mathbf{Q}})$, then by Proposition 2.4.4 [2] the total height given by (2.1) satisfies

$$h_{\mathcal{A}_L}(P) = h(P) \tag{4.8}$$

where the height on the right-hand side is the projective height of $P \in \mathbf{P}^n(\overline{\mathbf{Q}})$. Let $X_0, \ldots, X_n$ denote the projective coordinates on $\mathbf{P}^n$. Throughout this proof $c_1, c_2, \ldots$ denote positive constants which are independent of N if not stated otherwise.

Let C be the curve from Proposition 3.1. Any point $Q \in C$ is a non-singular point of C and of $\mathcal{A}_L$. By Example II 8.22.1 [9] there is a Zariski open neighborhood V of Q in $\mathcal{A}_L$ and homogeneous polynomials $H_1, \ldots, H_g \in \overline{\mathbf{Q}}[X_0, \ldots, X_n]$ with $\deg H_1 = \cdots = \deg H_g \leq c_1$ such that

$$C \cap V \text{ is cut out on } V \text{ by } H_1, \ldots, H_g.$$

We fix $H_0 \in \overline{\mathbf{Q}}[X_0, \ldots, X_n]$ such that $H_0(Q) \neq 0$ and $\deg H_0 = \deg H_1$. We replace V by a possibly smaller neighborhood of Q on which H_0 does not vanish. By quasi-compactness, C can be covered by c_2 such V and c_1 is independent of V.

Let $c_3 = c > 0$ be from Proposition 3.1 and N an integer with $2^N \geq c_3^{-1}$. The proposition gives us at least $c_3(2^N)^{2g} = c_3 4^{gN}$ distinct points $P \in X(\mathbf{C})$ such that

$$\left\{[2^N](P)\right\} \quad \text{is an irreducible component of} \quad [2^N](X) \cap C. \tag{4.9}$$

By the Pigeonhole Principle and after replacing c_3 by c_3/c_2 we may assume that all P as above satisfy $[2^N](P) \in C \cap V$, where V is among the fixed Zariski open sets from the covering above. After replacing c_3 by $c_3/(n+1)$, we may suppose that some fixed coordinate of all P is non-zero. These P are then contained in a non-empty Zariski open subset of $\mathcal{A}_L$ on which the Segre morphism can be inverted using monomials.

We use Lemma 4.3 to see that $[2^N]$ equals $[G_0 : \cdots : G_n]$ on a Zariski open and non-empty subset of $\mathcal{A}_L$ where $G_i \in \mathbf{Z}[X_0, \ldots, X_n]$ have suitably bounded degree. Let $H_0, \ldots, H_g$ be the polynomials attached to V. We set $F_i = H_i(G_0, \ldots, G_n) \in \overline{\mathbf{Q}}[X_0, \ldots, X_n]$ for $0 \leq i \leq g$. The F_i are homogeneous with $\deg F_i \leq c_4 4^N$.

The rational map $f : X \dashrightarrow \mathbf{P}^g$ given by $f = [F_0 : \cdots : F_g]$ is regular at the points P considered above; indeed, by construction $[G_0(P) : \cdots : G_n(P)] = [2^N](P)$ lies in $V(\mathbf{C})$ and is thus not a zero of H_0. Moreover, we have $f(P) = [1 : 0 : \cdots : 0]$. We claim that each P is an irreducible component of $f^{-1}([1 : 0 : \cdots : 0])$. We aim for a contradiction by assuming that there is an irreducible curve $Y \subset \mathrm{dom}(f)$ containing P with $f(Y) = [1 : 0 : \cdots : 0]$. Without loss of generality we may assume $Y \subset [2^N]^{-1}(V)$. But then $[2^N](Y) \subset V$ is in the set of common zeros of $H_1, \ldots, H_g$, hence $[2^N](Y) \subset [2^N](X) \cap C$. Now $[2^N](Y)$ remains a curve and it contains $[2^N](P)$. But this contradicts (4.9).

So the fiber $f^{-1}([1 : 0 : \cdots : 0])$ contains at least $c_3 4^{gN}$ isolated points. By Lemma 4.1 we conclude $\deg f \geq c_3 4^{Ng}$.

The proposition will now follow from Lemma 4.2 applied to the Zariski closure of X in $\mathbf{P}^n$. Indeed, taking $D = c_4 4^N$ we get $h(f(P)) \geq c_5 4^N h(P) - c_6(N)$ for all $P \in U(\overline{\mathbf{Q}})$ where $U \subset X$ is Zariski open and dense and $c_6(N)$ is a constant which may depend on

N. From (4.8) we conclude $h(f(P)) \geq c_5 4^N h_{\mathcal{A}_L}(P) - c_6(N)$. After shrinking U we may assume that $U \subset V$. Hence $f(P) = H([2^N](P))$ with $H = [H_0 : \cdots : H_g]$ and thus

$$h(H([2^N](P))) \geq c_5 4^N h_{\mathcal{A}_L}(P) - c_6(N). \tag{4.10}$$

Using the local definition of the projective height given in Chapter 1.5 [2] together with the triangle, respectively the ultrametric inequality we find $c_7 \geq 0$ such that

$$h(H(P')) \leq c_7 \max\{1, h(P')\}$$

for all $P' \in \mathbf{P}^n(\overline{\mathbf{Q}})$ such that at least one $H_i(P') \neq 0$. If $P' = [2^N](P)$ we obtain $h(H([2^N](P))) \leq c_7 \max\{1, h([2^N](P))\}$. But $h([2^N](P)) = h_{\mathcal{A}_L}([2^N](P))$ by (4.8). The proposition now follows from (4.10). $\square$

5. Passing to the Néron-Tate Height and Proof of the Main Results

5.1. **A Weak Version of Theorem 1.3 for the Legendre Family.** The height involved in the upper bound of Proposition 4.1 is the total height given by (2.1). On a fixed fiber of $\mathcal{A}_L$ above $Y(2)(\overline{\mathbf{Q}})$ this height differs from the Néron-Tate height (2.2) by a bounded function. The dependency of this bound on the fiber was made explicit by Silverman-Tate [22] and Zimmer [28]. See also the related result of Zarhin and Manin [27].

Theorem 5.1. *There is an absolute constant $c > 0$ such that if $P \in \mathcal{A}_L(\overline{\mathbf{Q}})$, then*

$$|h_{\mathcal{A}_L}(P) - \hat{h}_{\mathcal{A}_L}(P)| \leq c \max\{1, h(\pi_L(P))\}.$$

Proof. This follows from either Zimmer's Theorem or from the result of Silverman and Tate, cf. Theorem A [22]. $\square$

On X we can now bound $h(\pi_L(P))$ from above in terms of $\hat{h}_{\mathcal{A}_L}(P)$ by applying Proposition 4.1 to a sufficiently large but fixed integer N.

Lemma 5.1. *Let $X \subset \mathcal{A}_L$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$ which dominates $Y(2)$, has dimension g, and satisfies (i) or (ii) of Proposition 3.1. There exist a constant $c = c(X) > 0$ and a non-empty Zariski open subset $U \subset X$ such that*

$$h(\pi_L(P)) \leq c \max\{1, \hat{h}_{\mathcal{A}_L}(P)\}$$

for all $P \in U(\overline{\mathbf{Q}})$.

Proof. Let $c_1 > 0$ be the constant c in Proposition 4.1 and let c_2 be c from Theorem 5.1. We fix N to be the least positive integer with $2^N \geq \max\{2^{1/c_1}, \sqrt{2c_2/c_1}\}$. Proposition 4.1 implies

$$c_1 4^N h_{\mathcal{A}_L}(P) - c_3(N) \leq h_{\mathcal{A}_L}([2^N](P))$$

for all $P \in U(\overline{\mathbf{Q}})$ where $c_3(N)$ is $c'(N)$ from said proposition. We use the bound for $|h_{\mathcal{A}_L}(P) - \hat{h}_{\mathcal{A}_L}(P)|$ to obtain the second inequality in

$$c_1 4^N h(\pi_L(P)) - c_3(N) \leq c_1 4^N h_{\mathcal{A}_L}(P) - c_3(N) \leq \hat{h}_{\mathcal{A}_L}([2^N](P)) + c_2 \max\{1, h(\pi_L(P))\},$$

the first one follows from (2.1).

The Néron-Tate height is quadratic. Dividing by 4^N leads to

$$\left(c_1 - \frac{c_2}{4^N}\right) h(\pi_L(P)) \leq \hat{h}_{\mathcal{A}_L}(P) + \frac{c_3(N) + c_2}{4^N}.$$

By our choice of N we have $c_1 - c_2/4^N \geq c_1/2 > 0$ and the current lemma follows. □

The previous lemma only holds for hypersurfaces in $\mathcal{A}_L$. Moreover, the restriction (i) in Proposition 3.1 is stronger than $X^\star \neq \emptyset$, the implicit condition of Theorem 1.3(ii). We address both issues in the next lemma which is essentially an induction on dimension.

Lemma 5.2. *Let $X \subset \mathcal{A}_L$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$ which is not an irreducible component of a flat subgroup scheme of $\mathcal{A}_L$. There is a constant $c = c(X) > 0$ and a Zariski open non-empty set $U \subset X$ such that*

$$h(\pi_L(P)) \leq c \max\{1, \hat{h}_{\mathcal{A}_L}(P)\} \tag{5.1}$$

for all $P \in U(\overline{\mathbf{Q}})$.

Proof. The hypothesis implies $X \neq \mathcal{A}_L$. The lemma certainly holds for all X which do not dominate $Y(2)$ because $\pi_L|_X$ is then constant. Hence we reduce to the case where X dominates $Y(2)$. The proof is by induction on $\dim X + \dim \mathcal{A}_L = \dim X + g + 1$. The induction parameter is at least 3.

If $\dim X + \dim \mathcal{A}_L = 3$, then the only possibility is $\dim X = 1$ and $\dim \mathcal{A}_L = 2$, so $\mathcal{A}_L = \mathcal{E}_L$. By hypothesis, we are in case (ii) of Proposition 3.1. The height inequality (5.1) follows from Lemma 5.1.

So let us assume $\dim X + \dim \mathcal{A}_L \geq 4$. Because $\dim X < \dim \mathcal{A}_L$ we split into two cases.

The first case is when X is no hypersurface, so $\dim X \leq g - 1$. We pick any $\lambda \in Y(2)(\overline{\mathbf{Q}})$. All irreducible components of the fiber X_λ have dimension at least $\dim X - 1$ by the Fiber Dimension Theorem. But since X dominates $Y(2)$, all these irreducible components have dimension precisely $\dim X - 1$. Moreover, any X_λ is non-empty because $\pi_L|_X : X \to Y(2)$ is dominant and proper, hence surjective. Let $Z \subset X_\lambda$ be an irreducible component. The whole fiber $(\mathcal{A}_L)_\lambda = (\mathcal{E}_L)^g_\lambda$ is a power of an elliptic curve. So there is a projection $\Psi : (\mathcal{E}_L)^g_\lambda \to (\mathcal{E}_L)^{\dim X - 1}_\lambda$ onto $\dim X - 1$ coordinates such that $\dim Z = \dim \Psi(Z)$. We again apply the Fiber Dimension Theorem to find a point $P \in Z(\overline{\mathbf{Q}})$ with $\dim_P \Psi|_Z^{-1}(\Psi(P)) = 0$. We can even arrange that P is not contained in any other irreducible component of X_λ. Of course, Ψ extends to a projection $\mathcal{A}_L \to \mathcal{E}_L \times_{Y(2)} \cdots \times_{Y(2)} \mathcal{E}_L$ ($\dim X - 1$ factors). An irreducible component of $\Psi|_X^{-1}(\Psi(P)) \subset X_\lambda$ containing P must be $\{P\}$. A final application of the Fiber Dimension Theorem shows $\dim \Psi(X) \geq \dim X$. But the reverse inequality also holds. So $\dim \Psi(X) = \dim X$. To simplify notation we assume that Ψ projects onto the first $\dim X - 1$ coordinates. For $\dim X \leq j \leq g$ let $\Psi_j : \mathcal{A}_L \to \mathcal{E}_L \times_{Y(2)} \cdots \times_{Y(2)} \mathcal{E}_L = \mathcal{B}$ ($\dim X$ factors) be the projection onto the first $\dim X - 1$ and the j-th coordinate. We note $\dim \mathcal{B} = 1 + \dim X$. Then Ψ_j is proper and so $\Psi_j(X)$ is an irreducible closed subvariety of $\mathcal{B}$. It follows quickly that $\dim \Psi_j(X) = \dim X$, so $\Psi_j(X)$ has codimension 1. We claim that at least one $\Psi_j(X)$ is not contained in a proper flat subgroup scheme of $\mathcal{B}$. Indeed, otherwise X would be contained in a flat subgroup scheme of dimension $\dim X$ by Lemma 2.5(ii) and (iii). Hence X would be an irreducible component of a flat subgroup scheme. This is impossible by hypothesis. So let us assume that $X' = \Psi_j(X)$ is not contained in a proper flat subgroup scheme of $\mathcal{B}$. Since $X' \neq \mathcal{B}$ we conclude that X' is not an irreducible component of a flat subgroup scheme. Because $\dim X' + \dim \mathcal{B} = 2\dim X + 1 \leq \dim X + \dim \mathcal{A}_L - 1$ we may

apply induction. So there is $c_1 > 0$ and a non-empty Zariski open subset $U' \subset X'$ such that $h(\pi_L(Q)) \leq c_1(1 + \hat{h}_{\mathcal{B}}(Q))$ for all $Q \in U'(\overline{\mathbf{Q}})$. The current case follows with $U = \Psi_j|_X^{-1}(U')$ because $\pi_L(P) = \pi_L(\Psi_j(P))$ and $\hat{h}_{\mathcal{B}}(\Psi_j(P)) \leq \hat{h}_{\mathcal{A}_L}(P)$; in fact, we are just omitting certain coordinates.

The second case is when X is a hypersurface, so $\dim X = g$; it is here where we apply Lemma 5.1. We split up into two subcases. In the first subcase we suppose $\dim \varphi(X) \geq 2$ for all $\varphi \in \mathbf{Z}^g \smallsetminus \{0\}$. That is, X satisfies hypothesis (i) of Proposition 3.1. We conclude this subcase immediately by applying Lemma 5.1. It remains to treat the case where there exists $\varphi \in \mathbf{Z}^g \smallsetminus \{0\}$ such that $\dim \varphi(X) \leq 1$, in this case $C = \varphi(X)$ is an irreducible closed curve in $\mathcal{E}_L$. It satisfies property (ii) of Proposition 3.1. We note that $4 \leq \dim X + \dim \mathcal{A}_L$ hence $\dim C + \dim \mathcal{E}_L < \dim X + \dim \mathcal{A}_L$. By induction there is $c_2 > 0$ with $h_{\mathcal{E}_L}(Q) \leq c_2 \max\{1, \hat{h}_{\mathcal{E}_L}(Q)\}$ for all $Q \in C(\overline{\mathbf{Q}})$; indeed any non-empty Zariski open subset of C misses merely finitely many points of C. Writing $Q = \varphi(P)$ for some $P \in X(\overline{\mathbf{Q}})$ we see $h(\pi_L(P)) = h(\pi_L(\varphi(P))) \leq c_2 \max\{1, \hat{h}_{\mathcal{E}_L}(\varphi(P))\}$ for all $P \in X(\overline{\mathbf{Q}})$.

It is well-known how to bound $\hat{h}_{\mathcal{E}_L}(\varphi(P))$ from above in terms of $\hat{h}_{\mathcal{A}_L}(P)$. Indeed, say $\varphi = (a_1, \ldots, a_g)$. If $P_1, \ldots, P_g \in (\mathcal{E}_L)_\lambda(\overline{\mathbf{Q}})$ where $\lambda \in Y(2)(\overline{\mathbf{Q}})$, then

$$\begin{aligned}\hat{h}_{\mathcal{E}_L}([a_1](P_1) + \cdots + [a_g](P_g)) &\leq g(a_1^2\hat{h}_{\mathcal{E}_L}(P_1) + \cdots + a_g^2\hat{h}_{\mathcal{E}_L}(P_g)) \\ &\leq g \max\{a_1^2, \ldots, a_g^2\}\hat{h}_{\mathcal{A}_L}(P);\end{aligned}$$

this follows from the fact that the Néron-Tate height is a quadratic form and from the Cauchy-Schwarz inequality. So $\hat{h}_{\mathcal{E}_L}(\varphi(P)) \leq g \max\{a_1^2, \ldots, a_g^2\}\hat{h}_{\mathcal{A}_L}(P)$. We conclude $h(\pi_L(P)) \leq c_3 \max\{1, \hat{h}_{\mathcal{A}_L}(P)\}$ for all $P \in X(\overline{\mathbf{Q}})$ where c_3 is independent of P. □

5.2. **Adding Level Structure and Proof of Theorem 1.3.** Let $\mathcal{E}, \mathcal{A}, S$, and π be as in the introduction. We recall that the curve S is defined over $\overline{\mathbf{Q}}$. We fix an irreducible and non-singular projective curve $\overline{S}$ and assume S is Zariski open in $\overline{S}$. Let $\mathcal{L}$ be a line bundle on $\overline{S}$ and let $h_{\overline{S},\mathcal{L}}$ be a choice of height function $\overline{S}(\overline{\mathbf{Q}}) \to \mathbf{R}$.

Before coming to the proof of Theorem 1.3(ii) we need an auxiliary construction.

Lemma 5.3. *Let us assume that $\mathcal{A}$ is not isotrivial. After possibly replacing S by a non-empty Zariski open subset there exists an irreducible non-singular quasi-projective curve S' defined over $\overline{\mathbf{Q}}$ with the following property. We have a commutative diagram*

$$\begin{tikzcd} \mathcal{A} \arrow[d, "\pi"'] & \mathcal{A}' \arrow[l, "f"'] \arrow[r, "e"] \arrow[d] & \mathcal{A}_L \arrow[d, "\pi_L"] \\ S & S' \arrow[l, "l"] \arrow[r, "\lambda"'] & Y(2) \end{tikzcd} \tag{5.2}$$

where l is finite, λ is quasi-finite, $\mathcal{A}'$ is the abelian scheme $\mathcal{A} \times_S S'$, f is finite and flat, and e is quasi-finite and flat. Moreover, the restriction of f and e to any fiber of $\mathcal{A}' \to S'$ is an isomorphism of abelian varieties. Finally, if $P \in \mathcal{A}'_s(\overline{\mathbf{Q}})$, then

$$\hat{h}_{\mathcal{A}}(f(P)) = \hat{h}_{\mathcal{A}_L}(e(P)). \tag{5.3}$$

Proof. Let $j : S \to Y(1)$ be the morphism as before Lemma 2.4. It is non-constant by said lemma.

We regard j as an element of $\overline{\mathbf{Q}}(\overline{S})$, the function field of $\overline{S}$. We may fix a finite field extension K of $\overline{\mathbf{Q}}(S)$ such that the generic fiber of $\mathcal{E} \to S$ is isomorphism, over K, to an elliptic curve determined by $y^2 = x(x-1)(x-\lambda)$ with $\lambda \in K$. Then K is the function field of an irreducible non-singular projective curve $\overline{S}'$ and the inclusion $\overline{\mathbf{Q}}(S) \subset K$ induces a finite morphism $l : \overline{S}' \to \overline{S}$. The rational function $\lambda \in \overline{\mathbf{Q}}(\overline{S}')$ extends to a morphism $\lambda : \overline{S}' \to \mathbf{P}^1$.

We let S' denote the preimage of S in $\overline{S'}$. Hence we obtain a irreducible non-singular quasi-projective curve S' over $\overline{\mathbf{Q}}$ such that

$$\begin{tikzcd} S' \arrow[r, "\lambda"] \arrow[d, "l"'] & Y(2) \arrow[d, "\lambda \mapsto 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}"] \\ S \arrow[r, "j"'] & Y(1) \end{tikzcd}$$

commutes; by abuse of notation we use the symbols λ and l to denote their restrictions to S'. The restricted morphism $l : S' \to S$ remains finite. We see that λ is a non-constant morphism between two irreducible curves; hence it is quasi-finite. After replacing S and S' by Zariski open subsets we may assume that $\lambda : S' \to Y(2)$ is étale and $l : S' \to S$ is finite and étale. After shrinking S, we still have an abelian scheme $\mathcal{A} \to S$.

The fibered product $\mathcal{E}' = \mathcal{E} \times_S S'$ is an abelian scheme over S' with elliptic curves as fibers. By construction, the generic fiber of $\mathcal{E}' \to S'$ is isomorphic, over $\overline{\mathbf{Q}}(S')$, to the elliptic curve defined by $y^2 = x(x-1)(x-\lambda)$. By Proposition 8, page 15 [3] an isomorphism on the generic fiber extends to an isomorphism on the whole abelian scheme. We remark that no argument in this paper relies on the existence of a Néron model. This provides the top arrow in the commutative square on the left in

$$\begin{tikzcd} \mathcal{E}' \arrow[r] \arrow[d] & S' \times_{Y(2)} \mathcal{E}_L \arrow[r] \arrow[d] & \mathcal{E}_L \arrow[d] \\ S' \arrow[r, equal] & S' \arrow[r, "\lambda"'] & Y(2) \end{tikzcd}$$

the square on the right is Cartesian.

Let $\mathcal{A}'$ be the g-fold fibered power of $\mathcal{E}'$ over S'. We remark that there is a natural morphism $\mathcal{A}' \to Y(2)$. We take the product over $Y(2)$ of the morphisms $\mathcal{A}' \to \mathcal{E}' \to \mathcal{E}_L$ coming from the g projections to get the right square in (5.2), the square on the left is the product over S of the g morphisms $\mathcal{A}' \to \mathcal{E}' \to \mathcal{E}$.

We claim that e is flat. By Corollary 11.3.11 [7, EGA IV$_3$] it suffices to prove that the following statement. Say x is in $Y(2)$ below s, a point of S'. We must show that e restricts to a flat morphism $\mathcal{A}'_x \to (\mathcal{A}_L)_x$ where $\mathcal{A}'_x$ and $(\mathcal{A}_L)_x$ are the fibers of $\mathcal{A}' \to Y(2)$ and $\mathcal{A}_L \to Y(2)$ above x, respectively. We consider the scheme theoretic fiber $\lambda^{-1}(x) \to \operatorname{Spec} k(x)$. Since $S' \to Y(2)$ is étale, $\lambda^{-1}(x)$ is étale over $\operatorname{Spec} k(x)$. We have a natural morphism $\operatorname{Spec} k(s) \to \lambda^{-1}(x)$ which when composed with $\lambda^{-1}(x) \to$

$\operatorname{Spec} k(x)$ is the étale morphism $\operatorname{Spec} k(s) \to \operatorname{Spec} k(x)$. So $\operatorname{Spec} k(s) \to \lambda^{-1}(x)$ is étale, cf. Corollary 17.3.5 [7, EGA IV$_4$] , and in particular flat. The induced morphism $\mathcal{A}'_x \times_{\lambda^{-1}(x)} \operatorname{Spec} k(s) \to \mathcal{A}'_x$ is flat too; but this new fibered product is $\mathcal{A}'_s$. Now the composition $\mathcal{A}'_s \to \mathcal{A}'_x \to (\mathcal{A}_L)_x$ is an isomorphism of abelian varieties over $k(x)$ and therefore flat. Corollary 2.2.11(iv) [7, EGA IV$_2$] implies that $\mathcal{A}'_x \to (\mathcal{A}_L)_x$ is flat at all points in the image of $\mathcal{A}'_s$. If we let s run over all points in the fiber of $S' \to Y(2)$ above x we conclude that $\mathcal{A}'_x \to (\mathcal{A}_L)_x$ is flat. In a similar way one can show that f is flat.

Since l and $\mathcal{A} \to S$ are proper, we see that $\mathcal{A}' \xrightarrow{f} \mathcal{A} \xrightarrow{\pi} S$ is proper by (5.2). Therefore, f is proper. We have $\dim \mathcal{A} = \dim \mathcal{A}' = \dim \mathcal{A}_L = g+1$ and Corollary III 9.6 [9] implies that f and e are quasi-finite. So f is finite.

By construction, the restriction of f and e to a fiber of $\mathcal{A}' \to S'$ determines an isomorphism of abelian varieties.

For each $s' \in S'(\overline{\mathbf{Q}})$ we have isomorphisms $\mathcal{E}'_{s'} \to \mathcal{E}_{l(s')}$ and $\mathcal{E}'_{s'} \to (\mathcal{E}_L)_{\lambda(s')}$ of elliptic curves. Hence $\mathcal{E}_{l(s')}$ and $(\mathcal{E}_L)_{\lambda(s')}$ are isomorphic. By our definition made in the introduction we see that Néron-Tate height of a point in $\mathcal{E}_{l(s')}$ equals the Néron-Tate height of its image in $(\mathcal{E}_L)_{\lambda(s')}$. Passing to the product gives (5.3). □

Lemma 5.4. *Let us assume that $\mathcal{A}$ is not isotrivial. Let $X \subset \mathcal{A}$ be an irreducible closed subvariety defined over $\overline{\mathbf{Q}}$ which is not an irreducible component of a flat subgroup scheme of $\mathcal{A}$. There is a constant $c = c(X) > 0$ and a non-empty Zariski open subset $U \subset X$ such that*

$$h_{\overline{S},\mathcal{L}}(\pi(P)) \le c \max\{1, \hat{h}_{\mathcal{A}}(P)\} \tag{5.4}$$

for all $P \in U(\overline{\mathbf{Q}})$.

Proof. We keep the notation from the previous lemma. Without loss of generality we may replace S by the non-empty Zariski open subset given there. Using Corollary III 9.6 [9] and the fact that f is finite and flat we see that the irreducible components of $f^{-1}(X)$ have dimension $\dim X$. As f is an open and closed morphism, it is surjective. So $f^{-1}(X)$ is non-empty. We pick an irreducible component X'' of $f^{-1}(X)$ and define X' to be the Zariski closure of $e(X'')$ in $\mathcal{A}_L$. Since $e|_{X''}$ has finite fibers, the Fiber Dimension Theorem implies $\dim X' = \dim X'' = \dim X$.

We claim that X' is not an irreducible component of a subgroup scheme of $\mathcal{A}_L$. We assume the contrary and deduce a contradiction. Let $G \subset \mathcal{A}_L$ be a subgroup scheme whose irreducible components dominate $Y(2)$ and such that one of them is X'. Then G is equidimensional of dimension $\dim X$ by Lemma 2.5(i). Since e is flat and quasi-finite we may again conclude that $e^{-1}(G)$ is equidimensional of dimension $\dim G = \dim X$. We see that X'' is an irreducible component of $e^{-1}(X')$. The fact that f is finite implies that $f(e^{-1}(G))$ is equidimensional of dimension $\dim X$ and $\dim f(X'') = \dim X$. Since X is irreducible and because f is closed, we have $f(X'') = X$. We remark $f(X'') \subset f(e^{-1}(G))$, so X is an irreducible component of $f(e^{-1}(G))$. Latter is a subgroup scheme of $\mathcal{A}$ by Lemma 5.3, hence it remains to show that all irreducible components of $f(e^{-1}(G))$ dominate S. An irreducible component of $e^{-1}(G)$ has dimension $\dim G$. Its image under the quasi-finite morphism e is Zariski dense in some irreducible component of G. Therefore, any irreducible component of $e^{-1}(G)$ dominates S'. Its image under f dominates S since $l : S' \to S$ is dominant. From this we conclude that any irreducible component of $f(e^{-1}(G))$ dominates S. We have a contradiction.

Thus we may apply Lemma 5.2 to X' and obtain a Zariski open and non-empty $U' \subset X'$ on which the height inequality holds. Certainly, $e|_{X''}^{-1}(U')$ is Zariski open in X'' and non-empty. From above we have $f(X'') = X$, so $f(e|_{X''}^{-1}(U'))$ contains a non-empty Zariski open subset U of X.

We claim that (5.4) holds on $U(\overline{\mathbf{Q}})$. Let $P \in U(\overline{\mathbf{Q}})$ lie above $s \in S(\overline{\mathbf{Q}})$ and say $P'' \in e^{-1}|_{X''}(U')(\overline{\mathbf{Q}})$ with $f(P'') = P$ and $e(P'') = P' \in U'(\overline{\mathbf{Q}})$. If P'' and P' lie above $s'' \in S'(\overline{\mathbf{Q}})$ and $s' \in Y(2)(\overline{\mathbf{Q}})$, respectively, then chasing around (5.2) yields $\lambda(s'') = s'$ and $l(s'') = s$.

We know that $h(s') \leq c_1 \max\{1, \hat{h}_{\mathcal{A}_L}(P')\}$ for some constant $c_1 > 0$ which does not depend on P. We recall that h is the projective height on $\mathbf{P}^1(\overline{\mathbf{Q}})$. By height properties, $h \circ \lambda$ is a choice for a representative of $h_{\overline{S}',\lambda^*\mathcal{O}(1)}$; with this choice we have $h_{\overline{S}',\lambda^*\mathcal{O}(1)}(s'') = h(s')$. Now λ is finite and $\mathcal{O}(1)$ is ample, so $\lambda^*\mathcal{O}(1)$ is ample. Therefore, there is a positive integer a such that $\lambda^*\mathcal{O}(1)^{\otimes a} \otimes l^*\mathcal{L}^{\otimes(-1)}$ is ample. Functorial properties of the height imply that $ah_{\overline{S}',\lambda^*\mathcal{O}(1)} \geq h_{\overline{S},\mathcal{L}} \circ l - c_2$ on $\overline{S}'(\overline{\mathbf{Q}})$ for some constant c_2. On inserting s'' we find $h_{\overline{S},\mathcal{L}}(s) \leq c_3 \max\{1, \hat{h}_{\mathcal{A}_L}(P')\}$ for some constant $c_3 > 0$ which is independent of P. Finally, (5.3) implies $\hat{h}_{\mathcal{A}_L}(P') = \hat{h}_{\mathcal{A}}(P)$ and this completes the proof. □

Proof of Theorem 1.3(ii). We prove the height inequality in the assertion by induction on the dimension. The case of dimension 0 being trivial we assume $\dim X \geq 1$. If X is an irreducible component of a flat subgroup scheme, then $X^\star = \emptyset$ and there is nothing to prove. So we may assume the contrary. By Lemma 5.4 inequality (1.1) holds on $(X \smallsetminus Z)(\overline{\mathbf{Q}})$ for some proper Zariski closed subset $Z \subsetneq X$. Let $Z = Z_1 \cup \cdots \cup Z_r$ be the decomposition into irreducible components. It suffices to show (1.1) on all $Z_i(\overline{\mathbf{Q}}) \cap X^\star(\overline{\mathbf{Q}})$. Since $\dim Z_i \leq \dim X - 1$ we may do induction on the dimension. We obtain the desired inequality for all P in $(X \smallsetminus Z)(\overline{\mathbf{Q}}) \cup {Z_1}^\star(\overline{\mathbf{Q}}) \cup \cdots \cup {Z_r}^\star(\overline{\mathbf{Q}})$. This set contains $X^\star(\overline{\mathbf{Q}})$ by a formal argument using the definition of $X^\star$. □

The next lemma implies part (i) of Theorem 1.3.

Lemma 5.5. *Let $X \subset \mathcal{A}$ be an irreducible closed subvariety defined over $\mathbf{C}$. Then $X^\star$ is Zariski open in X and empty if and only if X is itself an irreducible component of a flat subgroup scheme of $\mathcal{A}$.*

Proof. Without loss of generality we may assume that X dominates S, otherwise $X^\star = X$ by definition.

To prove the lemma it suffices to show the following statement. There are at most finitely many irreducible subvarieties of X that are irreducible components of a flat subgroup scheme of $\mathcal{A}$ and maximal with this property.

Let Z be such a subvariety; it must dominate S. The generic fibers Z_η and X_η of $\pi|_Z$ and $\pi|_X$, respectively, are subvarieties of $\mathcal{A}_\eta$, the generic fiber of π. Let $Y_\eta \subset X_\eta$ be a further variety which is defined and irreducible over $\mathbf{C}(S)$ and with $Z_\eta \subset Y_\eta$. We also assume that Y_η is an irreducible component of an algebraic subgroup of $\mathcal{A}_\eta$. Therefore, it is an irreducible component of the kernel $\ker \Psi_\eta$ for an endomorphism Ψ_η of $\mathcal{A}_\eta$. By Proposition 8, page 15 [3] we may extend Ψ_η to an endomorphism Ψ of $\mathcal{A}$. Let Y be the Zariski closure of Y_η in $\mathcal{A}$. Then Y_η is the generic fiber of $\pi|_Y$ by Proposition 2.8.5 [7, EGA IV$_2$] and the comment after its proof. Then $Y \subset \ker \Psi$ and $Z \subset Y \subset X$. By comparing dimensions using the Fiber Dimension Theorem one shows that Y is an

irreducible component of $\ker \Psi$. Therefore, $Z = Y$ by maximality. So, $Z_\eta = Y_\eta$. We have just shown that Z_η is an irreducible subvariety of X_η which is an irreducible component of an algebraic subgroup of $\mathcal{A}_\eta$ and which is maximal with this property. By Raynaud's Theorem, the Manin-Mumford Conjecture, Z_η comes from a finite set of subvarieties of X_η. But Z is the Zariski closure of Z_η in $\mathcal{A}$ and we see that there are only finitely many such Z.

It follows that $X \smallsetminus X^\star$ is a finite union of irreducible components of flat subgroups schemes of $\mathcal{A}$. The second claim of the lemma follows too. □

5.3. **Special Points on $\mathcal{A}$ and Proof of Theorem 1.1.** We recall that the j-invariant of an elliptic curve with complex multiplication is an algebraic number. Therefore, it makes sense to speak of its Weil height.

The following result of Poonen is needed for the proof of Theorem 1.1.

Lemma 5.6 (Poonen)**.** *Let $T \in \mathbf{R}$. Up-to $\overline{\mathbf{Q}}$-isomorphism there are only finitely many elliptic curves over $\overline{\mathbf{Q}}$ with complex multiplication and whose j-invariant has Weil height at most T.*

Proof. This is Lemma 3 [20]. □

Proof of Theorem 1.1. We recall that $\mathcal{A}$ and S are defined over $\overline{\mathbf{Q}}$. By hypothesis $\mathcal{A}$ is not isotrivial. By Lemma 2.4 the morphism $j : S \to Y(1)$ which associates to $s \in S(\mathbf{C})$ the j-invariant of $\mathcal{E}_s$ is dominant.

We begin with the elementary "if" direction. Say there is $s \in S(\mathbf{C})$ such that X is an irreducible component of an algebraic subgroup of $\mathcal{A}_s$ and such that $\mathcal{A}_s$ has complex multiplication. The claim follows since the set of torsion points of an abelian variety lies Zariski dense. Now say X is an irreducible component of a flat subgroup scheme of $\mathcal{A}$. The set of $s \in S(\overline{\mathbf{Q}})$ such that $\mathcal{A}_s$ has complex multiplication is infinite, hence Zariski dense in S. For any such s, the fiber X_s is an algebraic subgroup of $\mathcal{A}_s$ and thus contains a Zariski dense set of torsion points. It follows that if Z is the Zariski closure of all special points in X, then $Z \cap X_s = X_s$ for infinitely many $s \in S(\overline{\mathbf{Q}})$. Therefore, $Z = X$ by a dimension argument.

We now prove the "only if" direction. Let us assume that X is an irreducible closed subvariety of $\mathcal{A}$ which contains a Zariski dense set of special points. Let us also assume that X is not an irreducible component of a flat subgroup scheme of $\mathcal{A}$.

If $s \in S(\mathbf{C})$ such that $\mathcal{A}_s$ has complex multiplication then so does $(\mathcal{E}_L)_s$ and it follows that $j(s)$ is algebraic. But since S is defined over $\overline{\mathbf{Q}}$ we see $s \in S(\overline{\mathbf{Q}})$. Moreover, any $P \in \mathcal{A}(\mathbf{C})$ which is a torsion point of $\mathcal{A}_{\pi(P)}$ and for which this fiber has complex multiplication must be algebraic. Hence X contains a Zariski dense set of algebraic points. It follows that X is defined over $\overline{\mathbf{Q}}$.

Let $\overline{S}$ be as before the statement of Theorem 1.3 and $\mathcal{L}$ and ample line bundle on $\overline{S}$. By part (i) of said theorem we see that $X^\star$ is non-empty and Zariski open in X. So there is Zariski dense subset of points in $X^\star(\overline{\mathbf{Q}})$ which are torsion in a fiber with complex multiplication. Let P be in this set and $s = \pi(P)$. The Néron-Tate height of P vanishes because this point is torsion. So $h_{\overline{S},\mathcal{L}}(s)$ is bounded from above independently of P by Theorem 1.3.

The curve $\overline{S}$ is projective and non-singular so j extends to a morphism $\overline{S} \to \mathbf{P}^1$. By properties of the height, the projective height on $\mathbf{P}^1(\overline{\mathbf{Q}})$ is a valid choice for the

representative of $h_{\mathbf{P}^1,\mathcal{O}(1)}$ and $h\circ j$ is a valid representative for $h_{\overline{S},j^*\mathcal{O}(1)}$. Since $\mathcal{L}$ is ample, there is a positive integer a such that $\mathcal{L}^{\otimes a}\otimes j^*\mathcal{O}(1)^{\otimes(-1)}$ is ample. Functorial properties of the height imply that $ah_{\overline{S},\mathcal{L}}-h_{\overline{S},j^*\mathcal{O}(1)}$ is bounded from below on $\overline{S}(\overline{\mathbf{Q}})$. It follows that $h(j(s))$ is bounded from above independently of P. Poonen's result implies that the set of possible j-invariants of $\mathcal{E}_s$ is finite. So there are only finitely many possible s. Hence the Zariski dense set of P we consider is in finitely many fibers of $\pi|_X : X\to S$. This means that X is contained in $\mathcal{A}_s$ for some $s\in S(\overline{\mathbf{Q}})$ and this fiber must have complex multiplication. We now regard X as a subvariety of the fixed abelian variety $\mathcal{A}_s$. By hypothesis X contains a Zariski dense set of torsion points. The classical Manin-Mumford conjecture implies that X is an irreducible component of an algebraic subgroup of $\mathcal{A}_s$. So X is as in case (i) of the definition of special subvarieties given before Theorem 1.1. ☐

5.4. **Proof of Theorem 1.2.** Theorem 1.2 is a consequence of Theorem 1.3 and a result of Szpiro and Ullmo which is encapsulated in the next lemma.

Any elliptic curve E over $\overline{\mathbf{Q}}$ has a semi-stable Faltings height $h_F(E)$ which depends only on its $\overline{\mathbf{Q}}$-isomorphism class, see §1 [24].

Lemma 5.7. *Let E be an elliptic curve defined over $\overline{\mathbf{Q}}$ without complex multiplication. There exists a constant $c=c(E)$ with the following property. If E' is an elliptic curve such that there exists a isogeny $E\to E'$ with cyclic kernel of cardinality N, then*

$$h_F(E')\geq h_F(E)+\frac{1}{2}\log(N)-c\log\log(3N).$$

Proof. Let E,E', and N be as in the hypothesis. For a prime p let e_p be the exponent of p in the factorization of N. Szpiro and Ullmo's Théorème 1.1 [25] implies

$$h_F(E')\geq h_F(E)+\frac{1}{2}\log N-\sum_{p|N}\frac{p^{e_p}-1}{(p^2-1)p^{e_p-1}}\log p-c_1 \tag{5.5}$$

where the constant c_1 may depend E, but not on E' or N. We have

$$\frac{p^{e_p}-1}{(p^2-1)p^{e_p-1}}\leq\frac{p^{e_p}}{p^2p^{e_p-1}/2}=\frac{2}{p}.$$

And by an elementary calculation

$$\sum_{p|N}\frac{p^{e_p}-1}{(p^2-1)p^{e_p-1}}\log p\leq 2\sum_{p|N}\frac{\log p}{p}\leq c_2\log\log(3N),$$

with c_2 absolute. The lemma follows from (5.5). ☐

Lemma 5.8. *Let E be an elliptic curve defined over $\overline{\mathbf{Q}}$ without complex multiplication and let $T\in\mathbf{R}$.*

(i) *Up-to $\overline{\mathbf{Q}}$-isomorphism there are only finitely many elliptic curves over $\overline{\mathbf{Q}}$ which are isogenous to E and which have Faltings height at most T.*
(ii) *Up-to $\overline{\mathbf{Q}}$-isomorphism there are only finitely many elliptic curves over $\overline{\mathbf{Q}}$ which are isogenous to E and whose j-invariant has Weil height at most T*

Proof. The second part of the lemma follows from the first since bounding the j-invariant of an elliptic curve amounts to bounding its Faltings height by Proposition 2.1 [24].

Let E' be an elliptic curve which is isogenous to E with $h_F(E') \leq T$. There exists an isogeny $E \to E'$ with cyclic kernel of cardinality N, say; for a proof we refer to Lemma 6.2 [14]. By Lemma 5.7 we see that N is bounded in terms of T. So there are only finitely many possibilities for the kernel of $E \to E'$. Hence up-to $\overline{\mathbf{Q}}$-isomorphism there are only finitely many possibilities for E'. □

Proof of Theorem 1.2. The proof runs along the lines of the proof of Theorem 1.1. If E does not have complex multiplication we use Lemma 5.8 instead of Lemma 5.6. □

6. An Instance of the Bogomolov Conjecture over Functions Fields

Let $K, \overline{K}, E$ be as in the hypothesis of Theorem 1.4. So K is the function field of an irreducible non-singular projective curve $\overline{S}$ defined over $\overline{\mathbf{Q}}$. By considering a Weierstrass model over K of E we see that E is the generic fiber of an abelian scheme $\mathcal{E} \to S$; here S is a sufficiently small Zariski open and dense subset of $\overline{S}$. As in the introduction, let $\mathcal{A}$ be the g-fold fibered power of $\mathcal{E}$ over S with $\pi : \mathcal{A} \to S$ the structural morphism. The condition that E has non-constant j-invariant implies that $\mathcal{A}$ is not isotrivial. We write A for the generic fiber of $\pi : \mathcal{A} \to S$; this is just the abelian variety E^g over K.

On $\overline{S}$ we fix an ample line bundle $\mathcal{L}$. We also choose a representative of the equivalence class of height functions associated to the pair $\overline{S}, \mathcal{L}$ and denote it by $h_{\overline{S},\mathcal{L}}$, cf. Section 2.1.

Any finite field extension K' of K is the function field of an irreducible non-singular projective curve $\overline{S}'$ defined over $\overline{\mathbf{Q}}$. The inclusion $K \subset K'$ induces a finite morphism $\rho : \overline{S}' \to \overline{S}$. We set $S' = \rho^{-1}(S)$ and regard K' as the function field of S'.

A point $x \in A(K')$ induces a rational map $\widetilde{x} : S' \dashrightarrow \mathcal{A}$ such that $\pi \circ \widetilde{x} = \rho$ on the domain of $\widetilde{x}$.

Recall that we defined the Néron-Tate height $\hat{h}_{\mathcal{A}}$ on $\mathcal{A}$ in Section 2.1. For any algebraic point $t \in S'(\overline{\mathbf{Q}})$ in the domain of $\widetilde{x}$ it makes sense to speak of $\hat{h}_{\mathcal{A}}(\widetilde{x}(t))$.

On E we have a symmetric and ample line bundle coming from the zero element of E considered as a Weil divisor. Taking the tensor product of the pull-backs coming from the g projections $E^g \to E$ determines a symmetric and ample line bundle on A. Since K is equipped with a product formula in the sense of Chapter 1.4 [2], we may associate to said line bundle a Néron-Tate height $\hat{h}_A$; cf. Chapter 9.2 of the same reference.

Of course, $\hat{h}_A$ need not equal the height appearing in Theorem 1.4. However, functorial properties of the Néron-Tate height imply the following statement. If $\hat{h}'_A$ is a Néron-Tate height on $A(\overline{K})$ coming from a symmetric and ample line bundle there exists $c > 0$ such that $\hat{h}_A \leq c\hat{h}'_A$. Therefore, it suffices to prove Theorem 1.4 with the fixed height function described above.

We can now state Silverman's Theorem [22] applied to our situation.

Theorem 6.1. *In the notation above, let $t_1, t_2, \ldots \in S'(\overline{\mathbf{Q}})$ be a sequence of points in the domain of $\widetilde{x}$ such that $\lim_{k\to\infty} h_{\overline{S},\mathcal{L}}(\rho(t_k)) = \infty$. Then*

$$\lim_{k\to\infty} \frac{\hat{h}_{\mathcal{A}}(\widetilde{x}(t_k))}{h_{\overline{S},\mathcal{L}}(\rho(t_k))} = \hat{h}_A(x).$$

Proof. We apply Silverman's Theorem B to $A_{K'}$, the base change of A to K'. From this we will see that the limit equality holds. Indeed, we have $\hat{h}_{A_{K'}} = [K' : K]\hat{h}_A$ on $A_{K'}(K')$. Moreover, Silverman's choice of height on $\overline{S'}(\overline{\mathbf{Q}})$ is asymptotically equal to $[K' : K]^{-1}h \circ \rho$ by functorial properties of the height. □

We now combine the upper bound from Theorem 1.3 with the conclusion of Theorem 6.1 which serves as a competing lower bound.

Proof of Theorem 1.4. We let $i : A \to \mathcal{A}$ be the natural morphism and let $\mathcal{X}$ be the Zariski closure of $i(X)$ in $\mathcal{A}$. Then $\mathcal{X}$ is irreducible. By Proposition 2.8.5 [7, EGA IV$_2$] , $\mathcal{X}$ is flat over S and satisfies $i^{-1}(\mathcal{X}) = X$. Hence X is the generic fiber of $\pi|_{\mathcal{X}} : \mathcal{X} \to S$, so the Fiber Dimension Theorem implies $\dim X = \dim \mathcal{X} - 1$.

We claim that $\mathcal{X}$ is not an irreducible component of a flat subgroup scheme of $\mathcal{A}$. Let us assume the converse, we will arrive at a contradiction. A repeated application of Lemma 2.5(ii) together appropriate projections to $\mathcal{E}^{g'}$ with $g' \le g$ gives us independent $\varphi_1, \ldots, \varphi_{g-\dim \mathcal{X}+1} \in \mathbf{Z}^g$ such that $\mathcal{X} \subset \ker(\varphi_1 \times_S \cdots \times_S \varphi_{g-\dim \mathcal{X}+1})$. This implies a similar inclusion on the generic fiber. The common kernel of $\varphi_1, \ldots, \varphi_{g-\dim \mathcal{X}+1}$ considered as homomorphisms $A = E^g \to E$ is an algebraic subgroup of dimension $\dim \mathcal{X} - 1 = \dim X$. So it contains X as an irreducible component, this is a contradiction.

From Theorem 1.3(i) we conclude that $\mathcal{X} \smallsetminus \mathcal{X}^\star$ is Zariski closed and proper in $\mathcal{X}$. Let $c > 0$ be as in part (ii) of this theorem. We claim that c^{-1} is a suitable choice for ϵ and that the preimage of $\mathcal{X} \smallsetminus \mathcal{X}^\star$ under the dominant morphism $i|_X : X \to \mathcal{X}$ is a suitable choice for Z. Indeed, let $x \in (X \smallsetminus Z)(K')$ where K' is a finite field extension of K contained in $\overline{K}$. As above, there is an irreducible non-singular projective curve S' over $\overline{\mathbf{Q}}$ with function field K', a finite morphism $\rho : S' \to S$, and a rational map $\tilde{x} : S' \dashrightarrow \mathcal{A}$ such that $\rho = \pi \circ \tilde{x}$ on the domain of $\tilde{x}$.

The Zariski closure $\mathcal{Y}$ of the image of $\tilde{x}$ in $\mathcal{A}$ is an irreducible subvariety of $\mathcal{A}$. It has dimension at most 1. But it must be a curve since ρ is dominant. We have $\mathcal{Y} \subset \mathcal{X}$. Finally, $\mathcal{Y} \cap \mathcal{X}^\star \neq \emptyset$. because $x \notin Z(\overline{K})$.

So $\mathcal{Y} \cap \mathcal{X}^\star$ is a quasi-projective curve which dominates S. Since $\mathcal{L}$ is ample there is a sequence of points $P_1, P_2, \ldots \in (\mathcal{Y} \cap \mathcal{X}^\star)(\overline{\mathbf{Q}})$ such that

$$\lim_{k\to\infty} h_{\overline{S},\mathcal{L}}(\pi(P_k)) = \infty.$$

For k large enough there is $t_k \in S(\overline{\mathbf{Q}})$ with $\tilde{x}(t_k) = P_k$. Then $\lim_{k\to\infty} h_{\overline{S},\mathcal{L}}(\rho(t_k)) = \infty$ because $\pi(P_k) = \rho(t_k)$.

Theorem 1.3(ii) implies

$$\hat{h}_{\mathcal{A}}(\tilde{x}(t_k)) = \hat{h}_{\mathcal{A}}(P_k) \ge c^{-1} h_{\overline{S},\mathcal{L}}(\pi(P_k)) = \epsilon h_{\overline{S},\mathcal{L}}(\rho(t_k))$$

for k large enough since the left-hand side will eventually be greater than 1. So

$$\liminf_{k\to\infty} \frac{\hat{h}_{\mathcal{A}}(\tilde{x}(t_k))}{h_{\overline{S},\mathcal{L}}(\rho(t_k))} \ge \epsilon.$$

By Silverman's Theorem this limes inferior is in fact a limes which equals $\hat{h}_A(x)$. We conclude $\hat{h}_A(x) \ge \epsilon$. The theorem follows because $\epsilon > 0$ was independent of x. □

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